RESULT 5: UNDER ASSUMPTION BLOCKS 1-6, THE MARXIAN PRICES ARE PROPORTIONAL TO THE TRUE EQUILIBRIUM PRICES IF AND ONLY IF LINEAR DEPENDENCE OF INDUSTRIES HOLDS.

We need to prove two implications hold under Assumption Blocks 1-6:

Marxian prices proportional to true equilibrium prices ⇒ linear dependence of industries

And

Linear dependence of industries ⇒ Marxian prices proportional to true equilibrium prices

To prove the first implication, assume that the Marxian prices \( q_i \) are proportional to the true equilibrium prices \( p_i \); that is, there exists a real number \( \alpha \geq 0 \) such that \( p_i = \alpha q_i \) for all \( i \). But in fact, \( \alpha > 0 \), for note that \( p_i > 0 \) in light of the rate of profit \( \pi \) being \( > 0 \) under Assumption Blocks 1,4,5, and 6, and that \( q_i > 0 \) in light of the constant capital \( C_i \) being \( > 0 \) and the variable capital \( V_i \) being \( \geq 0 \) for all \( i \) under Assumption Blocks 1-4. Therefore, if \( p \) is the vector of equilibrium prices and \( q \) is the vector of Marxian prices, we have \( p = \alpha q \) with \( \alpha > 0 \). The price-determining equations therefore imply:

\[
p = (1 + \pi)pM \\
\Rightarrow \alpha q = (1 + \pi)\alpha qM \tag{1a}
\]

\[
\Rightarrow q = (1 + \pi)qM, \tag{1c}
\]

where \( M \) is the input power matrix and we have used the fact that \( \alpha > 0 \) to cancel it from both sides of (1b). Recall that by definition \( q = (1 + \pi)(C + V) \), where \( C \) and \( V \) are the vectors of constant and variable capitals, respectively, of all goods. Substituting this into both sides of (1c) yields:

\[
(1 + \pi)(C + V) = (1 + \pi)(1 + \pi)(C + V)M \\
\Rightarrow C + V = (1 + \pi)(C + V)M \tag{2b}
\]

Now note that, by definition of the constant and variable capitals, we have:

\[
C + V = [C_1 \cdots C_n C_{n+1} \cdots C_m] + [V_1 \cdots V_n V_{n+1} \cdots V_m] \\
= [\sum_{j=1}^{n} \lambda_j a_{j1} \cdots \sum_{j=1}^{n} \lambda_j a_{jn} \cdots \sum_{j=1}^{n} \lambda_j a_{jn+1} \cdots \sum_{j=1}^{n} \lambda_j a_{jm}] \\
+ [\omega \Lambda_{II} B_1 \cdots \omega \Lambda_{II} B_n \omega \Lambda_{II} B_{n+1} \cdots \omega \Lambda_{II} B_m] \tag{3a}
\]

\[
[\Lambda_I col_1 A_I \cdots \Lambda_I col_n A_I \Lambda_I col_1 A_{II} \cdots \Lambda_I col_{m-n} A_{II}] \\
+ \omega \Lambda_{II} B [l_1 \cdots l_n l_{n+1} \cdots l_m] \tag{3b}
\]

\[
= \Lambda_I [col_1 A_I \cdots col_n A_I col_1 A_{II} \cdots col_{m-n} A_{II}] + \omega \Lambda_{II} B[L_I \ L_{II}] \tag{3c}
\]

\[
= \Lambda_I [A_I \ A_{II} + \Lambda_{II} \omega BL_I \ \omega BL_{II}] \tag{3d}
\]

\[
= [\Lambda_I A_I + \Lambda_{II} \omega BL_I \ \Lambda_I A_{II} + \Lambda_{II} \omega BL_{II}] \tag{3e}
\]

\[
= [\Lambda_I A_I \ \Lambda_{II}] \begin{bmatrix} A_I & A_{II} \\ \omega BL_I & \omega BL_{II} \end{bmatrix} \tag{3f}
\]
\[ \Lambda M = (1 + \pi)(C + V)M \]  

(3i)

where \( A_I \) is the capital input coefficient matrix for capital goods, \( A_{II} \) is the capital input matrix for wage and luxury goods, \( L_I \) is the labor input vector for capital goods, \( L_{II} \) is the labor input coefficient vector for wage and luxury goods, \( \Lambda_I \) is the vector of labor values of capital goods, and \( \Lambda \) is the vector of labor values of all goods. Substituting (3i) into the left side of (2b) yields:

\[ \Lambda^2 = (1 + \pi) + (\pi + 1) \]

(4)

But recalling that by decomposition of value, \( \Lambda = C + V + S \) where \( S \) is the vector of surplus values of all goods, (4) becomes:

\[
\begin{align*}
(C + V + S)M &= (1 + \pi)(C + V)M \\
\Rightarrow (C + V)M + SM &= (C + V)M + \pi(C + V)M \\
\Rightarrow SM &= \pi(C + V)M, 
\end{align*}
\]

(5a)

(5b)

(5c)

So that linear dependence of industries holds, as was to be shown. This proves the first implication.

To prove the second, assume linear dependence of industries holds, so that \( SM = \pi(C + V)M \). Then reversing the steps from (4) to (5c), we see that \( \Lambda M = (1 + \pi)(C + V)M \). But using \( C + V = \Lambda M \), this becomes \( C + V = (1 + \pi)(C + V)M \). Multiplying both sides by \( (1 + \pi) \) then gives \( (1 + \pi)(C + V) = (1 + \pi)(1 + \pi)(C + V)M \). Using \( q = (1 + \pi)(C + V) \), we obtain \( q = (1 + \pi)qM \), or \( qM = \mu q \), where \( \mu = (1 + \pi)^{-1} \) is the Perron-Frobenius eigenvalue of \( M \). This shows that \( q \) is a left eigenvector associated with \( \mu \). But \( p \) is also a left eigenvector for \( \mu \) since \( pM = \mu p \) under the price-determining equations. The Perron-Frobenius Theorem implies that the left eigenspace of \( \mu \) has dimension 1, which means any two left eigenvectors for \( \mu \) must be constant multiples of each other. Thus, there exists a real number \( \alpha \) such that \( q = \alpha p \), which is to say that the Marxian prices are proportional to the true equilibrium prices, as was to be shown. This proves the second implication and the proof is now complete.