LEMMA 1: UNDER ASSUMPTION BLOCKS 1, 5, AND 6, THE INPUT POWER MATRIX $M$ IS PRODUCTIVE.

Under Assumption Blocks 1 and 5, $M$ is a non-negative, indecomposable matrix. By the Perron-Frobenius Theorem, it has a positive\(^1\) real eigenvalue $\lambda$ (called the “Perron-Frobenius eigenvalue) such that $\lambda$ has the largest absolute value among all eigenvalues of $\lambda$. Thus $\lambda$ is the spectral radius of $M$; under Assumption Block 5 we have $\lambda \leq 1$. Furthermore, $\lambda$ has a strictly positive eigenvector $x$ associated with it. Therefore, we have $0 < \lambda \leq 1$ and $x > 0$ such that:

$$Mx = \lambda x \quad (1)$$

Under Assumption Block 6, we have $\lambda < 1$, so that $\lambda x < x$ in light of $x > 0$. But then $Mx < x$, so that:

$$x - Mx > 0 \quad (2a)$$
$$\Rightarrow (I - M)x > 0 \quad (2b)$$

Since we have a strictly positive vector $x$ that yields strictly positive net output $(I - M)x$, we can conclude that $M$ is productive, and the result is proved.

---

\(^1\) The Perron-Frobenius eigenvalue $\lambda$ has to be positive. Clearly it is non-negative because it is the maximum absolute value among all eigenvalues of $M$. If it were 0, then we would have $Mx = 0$ with $M \geq 0$ and $x > 0$. The only way this could occur is if $M = 0$. But this would contradict the indecomposability of $M$. Thus $\lambda > 0$. 