MORISHIMA-SETON-O'KISHIO THEOREM: UNDER ASSUMPTION BLOCKS 1-5, IF THE RATE OF EXPLOITATION IS POSITIVE, IT IS STRICTLY GREATER THAN THE RATE OF PROFIT.

Recall from “Existence of Capitalist Equilibrium” that under Assumption Blocks 1, 4, and 5, it is possible to identify a unique non-negative rate of profit \( \pi \) and a unique strictly positive price vector \( p \) such that the price-determining equations \( p = (1 + \pi)pM \) are satisfied, where \( M \) is the input power matrix and the wage is the cost of the hourly subsistence bundle of goods \( \omega B \).

Recall also from equation (3c) in Result 2 that the rate of exploitation \( e \) satisfies:

\[
(1 + e)\omega \Lambda_{II}B = 1, \tag{1}
\]

where \( \omega = 1/T \) is the fraction of the workday \( T \) taken up by an hour, \( \Lambda_{II} \) is the vector of labor values of wage and luxury goods, and \( B \) is the vector of subsistence quantities of wage and luxury goods. Since (1) gives us an alternative way of writing the number “1”, we can replace any variable with itself times this number. This means we can write the labor value equations for capital goods and wage/luxury goods respectively as (see equations (8a) and (8b) in the “LTV Model Setup” document in “Morishima’s Labor Theory of Value Results”):

\[
\begin{align*}
\Lambda_{I} &= \Lambda_{I}A_{I} + (1 + e)\omega \Lambda_{II}BL_{I} \tag{2a} \\
\Lambda_{II} &= \Lambda_{I}A_{II} + (1 + e)\omega \Lambda_{II}BL_{II} \tag{2b}
\end{align*}
\]

By Result 6 in “Morishima’s Labor Theory of Value Results”, the labor values of all goods are strictly positive under Assumption Blocks 1-3. Thus, \( \Lambda_{I} \) in particular is a strictly positive vector. Furthermore, since under Assumption Block 2 every column of \( A_{I} \) has at least one positive entry, as does every column of \( A_{II} \), it follows that \( \Lambda_{I}A_{I} \) and \( \Lambda_{I}A_{II} \) are strictly positive vectors. Since we assume \( e > 0 \), we have \( e\Lambda_{I}A_{I} > 0 \) and \( e\Lambda_{I}A_{II} > 0 \). From (2a) and (2b) it now follows that:

\[
\begin{align*}
\Lambda_{I} < \Lambda_{I}A_{I} + e\Lambda_{I}A_{I} + (1 + e)\omega \Lambda_{II}BL_{I} \tag{3a} \\
\Rightarrow \Lambda_{I} < (1 + e)\Lambda_{I}A_{I} + (1 + e)\omega \Lambda_{II}BL_{I} \tag{3b} \\
\Rightarrow \Lambda_{I} < (1 + e)[\Lambda_{I}A_{I} + \omega \Lambda_{II}BL_{I}] \tag{3c}
\end{align*}
\]

and

\[
\begin{align*}
\Lambda_{II} < \Lambda_{I}A_{II} + e\Lambda_{I}A_{II} + (1 + e)\omega \Lambda_{II}BL_{II} \tag{4a} \\
\Rightarrow \Lambda_{II} < (1 + e)\Lambda_{II}A_{II} + (1 + e)\omega \Lambda_{II}BL_{II} \tag{4b} \\
\Rightarrow \Lambda_{II} < (1 + e)[\Lambda_{II}A_{II} + \omega \Lambda_{II}BL_{II}] \tag{4c}
\end{align*}
\]

Stacking the vector inequalities (3c) and (4c) into one big \((1 \times m)\) vector inequality yields:

\[
\begin{align*}
\begin{bmatrix} \Lambda_{I} & \Lambda_{II} \end{bmatrix} \begin{bmatrix} A_{I} & A_{II} \\ \omega B_{I} & \omega B_{II} \end{bmatrix} &= \begin{bmatrix} (1 + e)[\Lambda_{I}A_{I} + \omega \Lambda_{II}BL_{I}] & (1 + e)[\Lambda_{II}A_{II} + \omega \Lambda_{II}BL_{II}] \end{bmatrix} \tag{5a} \\
\Rightarrow \begin{bmatrix} \Lambda_{I} & \Lambda_{II} \end{bmatrix} < (1 + e)\begin{bmatrix} \Lambda_{I} & \Lambda_{II} \end{bmatrix} \begin{bmatrix} A_{I} & A_{II} \\ \omega B_{I} & \omega B_{II} \end{bmatrix} &\begin{bmatrix} A_{I} & A_{II} \\ \omega B_{I} & \omega B_{II} \end{bmatrix} \tag{5b} \\
\Rightarrow \begin{bmatrix} \Lambda_{I} & \Lambda_{II} \end{bmatrix} < (1 + e)[\begin{bmatrix} \Lambda_{I} & \Lambda_{II} \end{bmatrix}] &\begin{bmatrix} A_{I} & A_{II} \\ \omega B_{I} & \omega B_{II} \end{bmatrix} \tag{5c} \\
\Rightarrow \Lambda < (1 + e)\Lambda M, \tag{5d}
\end{align*}
\]
where $\Lambda = [A_1 \ A_H]$ is the vector of labor values of all goods.

Observe that there can be no non-negative, non-zero column vector $x$ such that $x = (1 + f)Mx$ for some $f \geq e$, for if there were, then in light of (5d) and the fact that $\Lambda$ is a strictly positive vector (and hence $(1 + e)\Lambda M$ must be as well) it would also be the case that:

\[
\Lambda x < (1 + e)\Lambda Mx
\]
\[
\leq (1 + f)\Lambda Mx \quad \{\text{since } e \leq f \text{ and } \Lambda Mx \geq 0\}
\]
\[
= \Lambda(1 + f)Mx
\]
\[
= \Lambda x,
\]

which is a contradiction.

Now suppose it were the case that the rate of profit exceeds the rate of exploitation, i.e., $\pi \geq e$. Then by the above argument there can be no non-negative, non-zero column vector $x$ such that $x = (1 + \pi)Mx$. So let $x$ be an $(m \times 1)$ vector satisfying $x = (1 + \pi)Mx$. Clearly such an $x$ must exist; for example $x = 0$. But assume that there exists a non-zero such $x$. This $x$ cannot be non-negative, so it must have some negative elements. (If $x$ has no positive elements, then $-x$ is a non-zero, non-negative vector such that $-x = (1 + \pi)M(-x)$, which is not allowed. So $x$ must have some positive as well as some negative elements). Let the vector obtained by replacing all the negative components of $x$ with zeroes be denoted $x^*$. Then, letting the (row $i$, column $j$) entry of $M$ be denoted $u_{ij}$, the fact that $x = (1 + \pi)Mx$ implies that for each $i = 1, \ldots, m$ we have:

\[
x_i = (1 + \pi)[u_{i1}x_1 + \cdots + u_{im}x_m]
\]

(7)

If $x_i \geq 0$, then we don’t change it when we change all the negative entries of $x$ to zero in the process of constructing $x^*$, so the $i^{th}$ entry of $x^*$ is just $x_i$; that is, $x_i^* = x_i$. So the left side of (7) does not change when we switch $x$ to $x^*$. However, the right-hand side either stays the same or increases, since $\pi > 0$ (in light of it being at least as large as $e$, which is assumed positive) and all the $u$‘s are $\geq 0$. Therefore, we have:

\[
x_i^* \leq (1 + \pi)[u_{i1}x_1^* + \cdots + u_{im}x_m^*]
\]

(8)

On the other hand, if $x_i < 0$, then in the process of creating $x^*$, $x_i$ gets switched to 0; that is, $x_i^* = 0$. But then certainly $x_i^*$ will be less than or equal to the right-hand side of (8) because $\pi > 0$, all the $u$‘s are $\geq 0$, and all the $x^*$‘s are $\geq 0$. Therefore, (8) actually holds for $i = 1, \ldots, m$, which, collecting all of these equations into an $(m \times 1)$ vector equation, implies:

\[
x^* \leq (1 + \pi)Mx^*
\]

(9)

But it must also be the case that $x^* \neq (1 + \pi)Mx^*$, since $x^*$ is a non-negative, non-zero vector and there is no such vector that satisfies $x^* = (1 + \pi)Mx^*$. Since $p > 0$, the fact that $x^* \leq (1 + \pi)Mx^*$, $x^* \neq (1 + \pi)Mx^*$, and $x^*$ has some strictly positive elements implies:

\[
p x^* < (1 + \pi)pMx^*
\]

(10)
But since \( p = (1 + \pi)pM \) as mentioned at the beginning of this document, post-multiplication by \( x^* \) yields:

\[
px^* = (1 + \pi)pMx^*,
\]

in contradiction of (10). Thus the assumption that there exists a non-zero \((m \times 1)\) column vector \( x \) such that \( x = (1 + \pi)Mx \) must be wrong; the only solution is \( x = 0 \). But in that case, the only solution to the matrix equation \([I - (1 + \pi)M]x = 0\) is the trivial solution. But that means the matrix \([I - (1 + \pi)M]\) is invertible. Applying this to the equation \( p = (1 + \pi)pM \), yields:

\[
\begin{align*}
p &= p(1 + \pi)M \\
\Rightarrow p[I - (1 + \pi)M] &= 0 \\
\Rightarrow p &= 0 \cdot [I - (1 + \pi)M]^{-1} = 0,
\end{align*}
\]

which contradicts the fact that \( p \) is strictly positive. Since we reach a contradiction no matter what we assume about the solution set to \( x = (1 + \pi)Mx \), the original assumption that \( \pi \geq e \) must have been wrong. We conclude that \( \pi < e \), so that if the rate of exploitation is positive, it must be strictly greater than the rate of profit, as was to be shown.

**DISCUSSION**

Marx anticipated this result in this next quote from Volume 3, keeping in mind that he equated the rate of surplus value to the rate of exploitation:

“‘It follows from this proportion that the rate of profit, \( p' \), is always smaller than \( s' \), the rate of surplus-value, because \( v \), the variable capital, is always smaller than \( C \), the sum of \( v + c \), or the variable plus the constant capital; the only, practically impossible case excepted, in which \( v = C \), that is, no constant capital at all, no means of production, but only wages are advanced by the capitalist.’” -Vol. 3, p. 33.

He derived this result from the equation \( p' = s' (v/C) = s' v/(c + v) \), which appears a few lines before the above quote in Volume III. We can see clearly from this equation that since in general \( v < c + v \), it will be the case that \( p' < s' \), unless it happens that \( c = 0 \) (a “practically impossible case” in Marx’s words), in which case \( p' = s' \). We don’t yet have this equation at our disposal (we will shortly) to use in establishing the result, since our rate of profit is expressed as the ratio of money terms instead of value terms. We also have to unpack the hidden assumptions on the input matrices and vectors that are needed in order to guarantee that the equation is true.