MORISHIMA-SETON EQUATION: UNDER ASSUMPTION BLOCKS 1-5, THE RATE OF PROFIT $\pi$ AND THE RATE OF EXPLOITATION $e$ ARE RELATED BY THE EQUATION $\pi = e \cdot \frac{V}{C + V}$, WHERE $C$ IS THE CONSTANT CAPITAL AND $V$ IS THE VARIABLE CAPITAL OF THE CHARACTERISTIC VECTOR OF GOODS.

Recall that in capitalist equilibrium with a subsistence wage, which exists under Assumption Blocks 1-5, we have $w = \omega p_B B$ and $p = (1 + \pi)p_M$, where $w$ is the wage, $\omega = 1/T$ is the fraction of the workday $T$ taken up by an hour, $B$ is the vector of subsistence quantities of wage and luxury goods, $p_B$ is the vector of prices of wage and luxury goods, $p > 0$ is the vector of prices of all goods, $\pi \geq 0$ is the rate of profit, and $M$ is the input power matrix.

Now we claim that there must exist a non-negative, non-zero vector $y$ such that $y = (1 + \pi)My$. Certainly there must exist some such vector because for example $y = 0$ will work. But $y = 0$ cannot be the only solution, for if it were, then the matrix $[I - (1 + \pi)M]$ would be invertible, but then the fact that $p[I - (1 + \pi)M] = 0$ would imply $p = 0$, contradicting the fact that $p > 0$. So there has to be a nonzero $y$ such that $y = (1 + \pi)My$. If $y$ has no negative components, then it’s clearly the vector we’re looking for. So let’s assume it has some negative components. If it has no positive components, then $-y$ is the vector we’re looking for because it’s a non-negative, non-zero vector such that $-y = (1 + \pi)M(-y)$. So let’s assume $y$ has some positive and some negative components. Then create a new vector $y^*$ by replacing the negative components of $y$ with 0 and keeping the non-negative components as they are. Then for the non-negative components the left-hand side of the equation $y = (1 + \pi)My$ remains the same, whereas the right-hand side can only get bigger or stay the same since all terms on the right are now non-negative and can only increase when we swap out the negative components of $y$ with 0. For the negative components, the left-hand side becomes 0 when we switch to $y^*$, but that’s still clearly no larger than the right-hand side, which consists of all non-negative terms. Thus, it follows that $y^* \leq (1 + \pi)My^*$. If $y^* = (1 + \pi)My^*$, then $y^*$ is the vector we’re looking for because it is non-negative (all negative terms have been switched to zero) and non-zero (it contains some positive terms). If $y^* \neq (1 + \pi)My^*$, then since $p > 0$ it follows from the non-negativity of both $y^*$ and $(1 + \pi)My^*$ along with the fact that each of them must have some strictly positive entries that $py^* < p(1 + \pi)My^*$. But, $p = (1 + \pi)pm$ implies $py^* = (1 + \pi)pMy^*$, a contradiction. Thus it must be the case that $y^* = (1 + \pi)My^*$ in the event that $y$ has some negative components. But in that case $y^*$ is the vector we’re looking for. And of course, if $y$ does not have any negative components, then it’s the vector we’re looking for. Either way, we have found the vector we’re looking for, i.e., there does exist a non-negative, non-zero vector $y$ such that:

$$y = (1 + \pi)My$$

(1)

Since $y$ is an $(m \times 1)$ vector (it has to be in order to be conformable for pre-multiplication by the $(m \times m)$ matrix $M$), let $y_I$ be the $(n \times 1)$ vector consisting of the first $n$ entries of $y$ and let $y_{II}$ be the $((m-n) \times 1)$ vector consisting of the next $m - n$ entries of $y$. Then we may write:

$$y = \begin{bmatrix} y_I \\ y_{II} \end{bmatrix}$$

(2)
We can identify the $n$ entries of $y_I$ with production amounts of each capital good and the $(m-n)$ entries of $y_{II}$ with production amounts of each wage/luxury good. We call $y$ the “characteristic vector of goods”, $y_I$ the “characteristic vector of capital goods,” and $y_{II}$ the “characteristic vector of wage and luxury goods.” Morishima also uses the terminology “balanced growth equilibrium” and “golden age equilibrium” in reference to $y$, since in the two-department model of extended reproduction it can be shown that when the economy is in a state of balanced growth, with the outputs of the two departments growing at the same rate (which is equal to the rate of profit), the departmental outputs are proportional to the combined labor values of the entries in $y_I$ and $y_{II}$.

Recall from the Morishima-Seton-Okishio Theorem that the labor value equations for capital and for wage/luxury goods can be respectively written:

$$\Lambda_I = \Lambda_I A_I + (1 + e)\omega \Lambda_{II} B L_I$$
$$\Lambda_{II} = \Lambda_I A_{II} + (1 + e)\omega \Lambda_{I} B L_{II},$$  \hspace{1cm} (3a)

where $\Lambda_I$ is the vector of labor values of capital goods, $\Lambda_{II}$ is the vector of labor values of wage and luxury goods, $A_I$ is the capital input matrix for capital goods, $A_{II}$ is the capital input matrix for wage and luxury goods, $L_I$ is the labor input vector for capital goods, $L_{II}$ is the labor input vector for wage and luxury goods, and $e$ is the rate of exploitation.

Post-multiply both sides of equation (3a) by $y_I$ and both sides of (3b) by $y_{II}$ to respectively get:

$$\Lambda_I y_I = \Lambda_I A_I y_I + (1 + e)\omega \Lambda_{II} B L_I y_I$$
$$\Lambda_{II} y_{II} = \Lambda_I A_{II} y_{II} + (1 + e)\omega \Lambda_{I} B L_{II} y_{II}$$ \hspace{1cm} (4a)

Now add (4a) and (4b) together to get:

$$\Lambda_I y_I + \Lambda_{II} y_{II} = \Lambda_I A_I y_I + (1 + e)\omega \Lambda_{II} B L_I y_I + \Lambda_I A_{II} y_{II} + (1 + e)\omega \Lambda_{I} B L_{II} y_{II}$$ \hspace{1cm} (5)

Pre-multiplying both sides of (1) by $\Lambda = [\Lambda_I \quad \Lambda_{II}]$, the vector of labor values of all goods, and then using the definitions of $\Lambda$, $y$, and $M$ gives:

$$\Lambda y = \Lambda(1 + \pi)M y$$
$$\Rightarrow \Lambda y = (1 + \pi)\Lambda M y$$ \hspace{1cm} (6a)

$$\Rightarrow [\Lambda_I \quad \Lambda_{II}] [y_I]^T = (1 + \pi)[\Lambda_I \quad \Lambda_{II}] M [y_I]^T$$ \hspace{1cm} (6b)

$$\Rightarrow [\Lambda_I \quad \Lambda_{II}] [y_I]^T = (1 + \pi)[\Lambda_I \quad \Lambda_{II}] \begin{bmatrix} A_I & A_{II} \\ \omega B L_I & \omega B L_{II} \end{bmatrix} [y_I]^T$$ \hspace{1cm} (6c)

$$\Rightarrow \Lambda_I y_I + \Lambda_{II} y_{II} = (1 + \pi)[\Lambda_I \quad \Lambda_{II}] \begin{bmatrix} \Lambda_I y_I + \Lambda_{II} y_{II} \\ \omega B L_I y_I + \omega B L_{II} y_{II} \end{bmatrix}$$ \hspace{1cm} (6d)

$$\Rightarrow \Lambda_I y_I + \Lambda_{II} y_{II} = (1 + \pi)[\Lambda_I \quad \Lambda_{II}] \begin{bmatrix} \Lambda_I y_I + \Lambda_{II} y_{II} \\ \omega B L_I y_I + \omega B L_{II} y_{II} \end{bmatrix}$$ \hspace{1cm} (6e)

$$\Rightarrow \Lambda_I y_I + \Lambda_{II} y_{II} = (1 + \pi)[\Lambda_I \quad \Lambda_{II}] \begin{bmatrix} \Lambda_I y_I + \Lambda_{II} y_{II} \\ \omega B L_I y_I + \omega B L_{II} y_{II} \end{bmatrix}$$ \hspace{1cm} (6f)

$$\Rightarrow \Lambda_I y_I + \Lambda_{II} y_{II} = (1 + \pi)[\Lambda_I A_I y_I + \Lambda_{II} A_{II} y_{II} + \Lambda_I \omega B L_I y_I + \Lambda_{II} \omega B L_{II} y_{II}]$$ \hspace{1cm} (6g)
Since the left-hand sides of (5) and (6g) are the same, the right-hand sides must be too, so we have:

\[ \Lambda_t A_I y_l + (1 + e) \omega \Lambda_{II} B L_I y_l + \Lambda_t A_{II} y_{II} + (1 + e) \omega \Lambda_{II} B L_{II} y_{II} = (1 + \pi) \Lambda_t A_I y_l + (1 + \pi) \Lambda_t A_{II} y_{II} + (1 + \pi) \Lambda_{II} \omega B L_I y_l + (1 + \pi) \Lambda_{II} \omega B L_{II} y_{II} \]

(7a)

\[ \Rightarrow e \omega \Lambda_{II} B L_I y_l + e \omega \Lambda_{II} B L_{II} y_{II} = \pi \Lambda_t A_I y_l + \omega \Lambda_{II} B L_I y_l + \pi \Lambda_t A_{II} y_{II} + \omega \Lambda_{II} B L_{II} y_{II} \]

(7b)

\[ \Rightarrow e(\omega \Lambda_{II} B L_I y_l + \omega \Lambda_{II} B L_{II} y_{II}) = \pi(\Lambda_t A_I y_l + \omega \Lambda_{II} B L_I y_l + \Lambda_t A_{II} y_{II} + \omega \Lambda_{II} B L_{II} y_{II}) \]

(7c)

\[ \Rightarrow e(\omega \Lambda_{II} B L_I y_l + \omega \Lambda_{II} B L_{II} y_{II}) = \pi(\Lambda_t A_I y_l + \Lambda_t A_{II} y_{II} + \omega \Lambda_{II} B L_I y_l + \omega \Lambda_{II} B L_{II} y_{II}) \]

(7d)

Consider the term \( \omega \Lambda_{II} B L_I y_l + \omega \Lambda_{II} B L_{II} y_{II} \), which appears on both sides of (7d). Since \( \omega \Lambda_{II} B \) is the labor value of an hour’s worth of subsistence goods, or the value of the goods required to power an hour’s worth of labor, and \( L_I y_l + L_{II} y_{II} \) is the number of labor hours required to produce the characteristic vector of goods, \( \omega \Lambda_{II} B (L_I y_l + L_{II} y_{II}) \) is the value of the subsistence goods needed to produce the labor for producing the characteristic bundle, or more simply, the value of the labor power needed to produce the characteristic bundle. And, consistent with our definition (see Result 2) of the variable capital of a good as the value of the labor power needed to produce a unit, we could call \( \omega \Lambda_{II} B (L_I y_l + L_{II} y_{II}) \) the variable capital of the characteristic output and denote it by \( V \).

Now consider the expression \( \Lambda_t A_I y_l + \Lambda_t A_{II} y_{II} \), which appears on the right side of (7d). The first term, \( \Lambda_t A_I y_l \), consists of the vector of labor values of the capital needed to produce a unit of each capital good, \( \Lambda_t A_I \), times the vector of capital goods in the characteristic capital output vector \( y_l \). Thus it measures the labor value of the capital needed to produce the characteristic output of capital goods. Similarly, \( \Lambda_t A_{II} y_{II} \) is the labor value of the capital needed to produce the wage and luxury goods in the characteristic vector. Therefore, the sum \( \Lambda_t A_I y_l + \Lambda_t A_{II} y_{II} \) is the value of the capital needed to produce the entire characteristic vector. Consistent with our definition in Result 2 of the constant capital of a good as the labor value of the capital needed to produce a unit, we can call \( \Lambda_t A_I y_l + \Lambda_t A_{II} y_{II} \) the constant capital of the characteristic output and denote it by \( C \). Using these definitions, we can now write (7d) as:

\[ eV = \pi(C + V) \]

(8)

Note that since \( y \) is a non-zero vector, either \( y_l \) or \( y_{II} \) must have a strictly positive entry. But \( \Lambda_t A_I \) and \( \Lambda_t A_{II} \) are both strictly positive vectors because \( \Lambda_t \) is strictly positive under Assumption Blocks 1-3 and each column of \( A_I \) has a strictly positive entry under Assumption Block 2, and the same for \( A_{II} \). It follows that \( C > 0 \). Since \( V \) is clearly non-negative, we have \( C + V > 0 \). We can therefore divide both sides of (8) by \( (C + V) \) to obtain:

\[ \pi = e \frac{V}{C+V}, \]

(9)

so that the Morishima-Seton equation holds and the result is proved.

Note that an implication of the Morishima-Seton equation is that a positive rate of exploitation is necessary for a positive rate of profit, since if \( e = 0 \) then \( \pi = 0 \) by (9). In other words, this equation essentially yields the Fundamental Marxian Theorem as an easy corollary. This raises
the question of why we didn’t just derive the FMT this way rather than proving it as a stand-alone result. The main reason is that we derived the FMT under weaker assumptions than the Morishima-Seton equation. We only needed Assumption Blocks 1-4 rather than 1-5, so we did not need to know that a capitalist equilibrium exists or that there is a uniform rate of profit across all industries. Rather, we were able to show that for all industries to earn profits simultaneously, even if those profits occurred at different rates, it was necessary for the rate of exploitation to be positive. The FMT that we proved is then a much stronger result than we can get as a corollary to the M-S equation. (A similar remark applies to deriving the FMT from the Morishima-Seton-Okishio Theorem, which also required Assumption Blocks 1-5).

It is also worth noting that in the above derivation of the Morishima-Seton equation, we assume that workers are paid just enough to afford the subsistence bundle of goods. So in effect they do not choose their bundle of goods, they are simply handed the subsistence basket. It turns out that the equation is still true when workers are allowed to choose the bundle of goods they wish to consume, so long as they all share the same preferences, as shown in the following corollary.

**MORISHIMA-SETON EQUATION WITH FLEXIBLE DEMANDS:** Suppose that workers have homogeneous non-satiated preferences and can choose their utility-maximizing bundles of goods rather than receiving the subsistence bundle. If assumption blocks 1-4 hold, and there exists a solution to the revised price-determining equations, then the Morishima-Seton equation still holds.

Assume workers are homogeneous in their preferences. The “non-satiated” aspect of preferences just means that more consumption is always preferred to less, so if the consumer has a positive amount of income, they will choose to spend all of it. Let $F$ be the vector of amounts of wage and luxury goods a worker chooses to consume when spending all of their daily wages. Note that as long as the wage is positive and all prices are positive, $F$ will only include finite non-negative entries, at least one of which will have to be strictly positive. The wage times the length of the work day should be the total cost of the bundle $F$, i.e.,

$$wT = p_{II}F$$  \hspace{1cm} (10)

Thinking of the demand vector $F$ as a function of the price vector $p_{II}$ and the daily wage $wT$, we may write $F = F(p_{II}, wT)$, so that (10) becomes:

$$wT = p_{II}F(p_{II}, wT)$$  \hspace{1cm} (11)

When the worker chooses goods by maximizing utility subject to their budget constraint, each element of the demand vector $F(p_{II}, wT)$ must be “homogeneous degree zero” in prices $p_{II}$ and daily wages $wT$. This means that if the prices of all goods and daily wages change by the same percentage, the quantity of each good demanded will not change, since the position of the budget constraint doesn’t change and obviously preferences have not changed. (So in the framework of microeconomic theory, the location of the tangency between the budget line and the highest attainable indifference curve cannot have changed). Note that if all prices and the wage change by the proportion $\alpha - 1$ (e.g., if $\alpha = 1.05$ we are saying that they change by .05 or 5%), then the new price vector is $\alpha p_{II}$ and the new daily wage is $\alpha wT$. By degree-0 homogeneity, these must result in the same demand vector as the worker was choosing with prices $p_{II}$ and daily wage $wT$. 

In other words, for any \( \alpha > 0 \), we have \( F(p_{II}, wT) = F(\alpha p_{II}, awT) \). In particular, \( F(p_{II}, wT) = F(\alpha p_{II}, awT) \) when \( \alpha = 1/w \), which implies:

\[
F(p_{II}, wT) = F\left(\frac{p_{II}}{w}, T\right) = F(p_{II}, w, T),
\]

where \( p_{II, w} = p_{II}/w \) is the vector of wage-prices of wage and luxury goods. Substituting (12b) into (11) yields:

\[
wT = p_{II}F(p_{II}, wT) \\
\Rightarrow T = \frac{p_{II}}{w} F(p_{II}, wT) \\
\Rightarrow T = p_{II, w}F(p_{II, w}, T) \\
\Rightarrow 1 = p_{II, w} \frac{F(p_{II, w}, T)}{T} \\
\Rightarrow 1 = p_{II, w}D,
\]

where \( D = F(p_{II, w}, T)/T \) is the hourly bundle of goods demanded. Note that, as long as \( T > 0 \), \( w > 0 \), and \( p_{II} > 0 \), \( D \) will be a finite, non-negative, non-zero vector of goods.

In this setup with utility-maximizing consumer choice, we replace the subsistence bundle \( B \) with the daily demand vector \( F(p_{II}, wT) \) in the unpaid-to-paid labor definition of the rate of exploitation (see equation (3) in Result 1), so that the definition becomes:

\[
e = \frac{T-A_{II}F(p_{II, w}, T)}{A_{II}F(p_{II, w}, T)}
\]

\[
= \frac{T\left(1-A_{II}\frac{F(p_{II, w}, T)}{T}\right)}{T\left(A_{II}\frac{F(p_{II, w}, T)}{T}\right)}
\]

\[
= \frac{1-A_{II}D}{A_{II}D}
\]

Note that (13e) implies that

\[
1 = \frac{p_{II}}{w} D \\
\Rightarrow w = p_{II}D,
\]

so that the wage will in fact be positive as long as \( p_{II} > 0 \) (which it will have to be in capitalist equilibrium) and \( D \) is non-zero. But recall that \( D \) being non-zero hinged on \( w \) being positive, so at this point the conditions are somewhat circular. They will be clarified shortly once we derive the vector form of the price-determining equations.

Substituting (15b) into the price-determining equations for capital goods and wage/luxury goods (see equations (7d) and (8d) in the Existence of Capitalist Equilibrium) yields:
\[ p_I = (1 + \pi)[p_I A_I + p_{II} DL_I] \]  
\[ p_{II} = (1 + \pi)[p_I A_{II} + p_{II} DL_{II}] \]  

Stacking (16a) and (16b) side-by-side into a \((1 \times m)\) vector equation, we get:

\[
\begin{bmatrix}
    p_I & p_{II}
\end{bmatrix} = \begin{bmatrix}
    (1 + \pi)[p_I A_I + p_{II} DL_I] & (1 + \pi)[p_I A_{II} + p_{II} DL_{II}]
\end{bmatrix}
\]

\[
\Rightarrow \begin{bmatrix}
    p_I & p_{II}
\end{bmatrix} = (1 + \pi) [p_I A_I + p_{II} DL_I] \begin{bmatrix}
    A_I & A_{II} \\
    DL_I & DL_{II}
\end{bmatrix}
\]

\[
\Rightarrow p = (1 + \pi) pN,
\]

where \(p = [p_I \ p_{II}]\) is the vector of prices of all goods and \(N = \begin{bmatrix}
    A_I & A_{II} \\
    DL_I & DL_{II}
\end{bmatrix}\) is the input power matrix in which labor power is determined by the hourly utility-maximizing bundle instead of the hourly subsistence bundle. Since \(D\) depends on \(p_{II}\) and \(w\), the input power matrix \(N\) does as well, so identifying a price vector \(p\), a wage \(w\), and a rate of profit \(\pi\) that solve the price determining equation (17d) is not as simple as making an assumption about the spectral radius of \(N\). If we write \(N = N(p, w)\), recognizing that \(N\) depends on \(p\) and \(w\), then \(N(p, w)\) is clearly a non-negative square matrix for all possible values of \(p\) and \(w\). If we were to also assume that for all values of \(p\) and \(w\) it is indecomposable, then the Perron-Frobenius Theorem would imply that it has a maximal eigenvalue with strictly positive eigenvector for each value of \(p\) and \(w\). But then in essence the candidate equilibrium price vector and rate of profit would depend on \(p\) and \(w\), not just the parameters of the model; we are no closer to solving the problem than before. Thus, this approach of identifying the maximal eigenvalue can’t give us equilibrium values of \(p, w\) and \(\pi\) that depend only on model parameters. Indeed, equation (17d) is potentially a highly non-linear equation in \(p, w\) and \(\pi\) depending on the form of the demand vector \(D\). Thus we will simply assume that a (unique) rate of profit \(\pi \geq 0\), price vector \(p > 0\), and wage \(w > 0\) can be found that satisfy (17d). Note that this takes care of the circularity problem of \(D\) being non-negative and \(w\) being positive each depending on the other. If we have found a solution to (17d) that involves a wage \(> 0\) while taking into account the form of \(D\), then we know the resulting \(D\) will necessarily have at least one non-zero element due to the non-satiation of preferences.

Now, using the exact same argument as in the case where workers simply receive the subsistence bundle every hour (replace \(M\) with \(N\) in that argument), there must exist a non-negative, nonzero vector \(y\) such that:

\[ y = (1 + \pi)Ny \]  

The vector \(y\) is the characteristic output vector for the case of flexible demands. We can again partition it into its first \(n\) entries, denoted \(y_I\), and its next \((m - n)\) entries, denoted \(y_{II}\). We then interpret \(y_I\) as the vector of characteristic outputs (or balanced growth equilibrium outputs) of capital goods and \(y_{II}\) as the vector of characteristic outputs of wage and luxury goods.

Note that (14c) implies \((1 + e)A_{II}D = 1\). We can use this to rewrite the labor value equations for capital goods and for wage/luxury goods respectively as:
\[ \Lambda_I = \Lambda_I A_I + (1 + e)\Lambda_{II} DL_I \]  
\[ \Lambda_{II} = \Lambda_I A_{II} + (1 + e)\Lambda_{II} DL_{II} \]  

By post-multiplying equation (19a) by \( y_I \) and (19b) by \( y_{II} \) we get:

\[ \Lambda_I y_I = \Lambda_I A_I y_I + (1 + e)\Lambda_{II} DL_I y_I \]  
\[ \Lambda_{II} y_{II} = \Lambda_I A_{II} y_{II} + (1 + e)\Lambda_{II} DL_{II} y_{II} \]  

Adding (20a) and (20b) together gives:

\[ \Lambda_I y_I + \Lambda_{II} y_{II} = \Lambda_I A_I y_I + (1 + e)\Lambda_{II} DL_I y_I + \Lambda_I A_{II} y_{II} + (1 + e)\Lambda_{II} DL_{II} y_{II} \]  

Pre-multiplying both sides of (18) by \( \Lambda = [\Lambda_I \quad \Lambda_{II}] \) and then using the definitions of \( y \) and \( N \) gives:

\[ \Lambda y = \Lambda (1 + \pi) N y \]  
\[ \Rightarrow \Lambda y = (1 + \pi) \Lambda N y \]  

\[ \Rightarrow [\Lambda_I \quad \Lambda_{II}] \begin{bmatrix} y_I \\ y_{II} \end{bmatrix} = (1 + \pi) [\Lambda_I \quad \Lambda_{II}] N \begin{bmatrix} y_I \\ y_{II} \end{bmatrix} \]  

\[ \Rightarrow [\Lambda_I \quad \Lambda_{II}] \begin{bmatrix} y_I \\ y_{II} \end{bmatrix} = (1 + \pi) [\Lambda_I \quad \Lambda_{II}] \begin{bmatrix} \Lambda_I y_I + \Lambda_{II} y_{II} \\ DL_I y_I + DL_{II} y_{II} \end{bmatrix} \]  

\[ \Rightarrow \Lambda_I y_I + \Lambda_{II} y_{II} = (1 + \pi) [\Lambda_I \quad \Lambda_{II}] \begin{bmatrix} \Lambda_I y_I + \Lambda_{II} y_{II} \\ DL_I y_I + DL_{II} y_{II} \end{bmatrix} \]  

Since the left-hand sides of (21) and (22g) are the same, the right-hand sides must be too, so we have:

\[ \Lambda_I A_I y_I + (1 + e)\Lambda_{II} DL_I y_I + \Lambda_I A_{II} y_{II} + (1 + e)\Lambda_{II} DL_{II} y_{II} = (1 + \pi) \Lambda_I A_I y_I + (1 + \pi) \Lambda_I A_{II} y_{II} + (1 + \pi) \Lambda_I A_{II} y_{II} + (1 + \pi) \Lambda_{II} DL_I y_I + (1 + \pi) \Lambda_{II} DL_{II} y_{II} \]  

\[ \Rightarrow e\Lambda_{II} DL_I y_I + e\Lambda_{II} DL_{II} y_{II} = \pi \Lambda_I A_I y_I + \pi \Lambda_{II} DL_I y_I + \pi \Lambda_I A_{II} y_{II} + \pi \Lambda_{II} DL_{II} y_{II} \]  

\[ \Rightarrow e(\Lambda_{II} DL_I y_I + \Lambda_{II} DL_{II} y_{II}) = \pi(\Lambda_I A_I y_I + \Lambda_{II} DL_I y_I + \Lambda_I A_{II} y_{II} + \Lambda_{II} DL_{II} y_{II}) \]  

The term \( \Lambda_{II} DL_I y_I + \Lambda_{II} DL_{II} y_{II} = \Lambda_{II} D(L_I y_I + L_{II} y_{II}) \) is the value of the bundle of goods that power the labor for producing the characteristic vector, or more simply, the value of the labor power that produces the characteristic output. (In this case, labor power is just defined by the hourly bundle of goods demanded rather than the hourly subsistence bundle). In essence it is then the variable capital of the characteristic output, which we denote \( V \). The term \( \Lambda_I A_I y_I + \Lambda_I A_{II} y_{II} \) is again the constant capital of the characteristic output and denoted \( C \). We can then write (23d) as:

\[ \Rightarrow eV = \pi(C + V) \]
Using the same argument as in the case of rigid demands, and noting that as explained above \( D \) is non-negative and non-zero, the fact that \( y \) is non-zero implies that \( C > 0 \) and hence \( C + V > 0 \). Dividing both sides of (24) by \((C + V)\), we obtain:

\[
\pi = e \frac{v}{c+v},
\]

(25)

and the Morishima-Seton equation once again holds.

**COROLLARY 1: (MORISHIMA-SETON-OKISHIO THEOREM) UNDER EITHER RIGID OR FLEXIBLE DEMANDS, THE RATE OF PROFIT IS STRICTLY LESS THAN THE RATE OF EXPLOITATION, UNLESS THE RATE OF EXPLOITATION IS ZERO, IN WHICH CASE THE RATE OF PROFIT IS ALSO ZERO.**

This follows immediately from either (9) or (25), since the fact that \( C + V > 0 \) under the assumptions of the Morishima-Seton equation means \( V/(C + V) \) will be in the interval \([0,1]\). In fact it is in the interval \([0,1)\) since \( C > 0 \). Thus if \( e > 0 \), we have \( 0 \leq \pi < e \), and if \( e = 0 \), then clearly \( \pi = 0 \) also. It is possible to have \( e > 0 \) and yet \( \pi = 0 \) in the event that \( V = 0 \), but even in this case it is still true that \( e > \pi \).

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**DISCUSSION**

As noted, this corollary is the Morishima-Seton-Okishio Theorem, slightly expanded to include the fact that the rate of exploitation and the rate of profit can be equal if there is no exploitation. Therefore, deriving the Morishima-Seton equation allows us to establish the Morishima-Seton-Okishio Theorem as an easy corollary. This is in fact the way Marx did it in Volume III:

“We retain the designations used in Books I and II. Total capital \( C \) consists of constant capital \( c \) and variable capital \( v \), and produces a surplus-value \( s \). The ratio of this surplus-value to the advanced variable capital, or \( s/v \), is called the rate of surplus-value and designated \( s' \). Therefore \( s/v = s' \), and consequently \( s = s'v \). If this surplus-value is related to the total capital instead of the variable capital, it is called profit, \( p \), and the ratio of the surplus-value \( s \) to the total capital \( C \), or \( s/C \), is called the rate of profit, \( p' \). Accordingly,

\[
p' = s/C = s/(c+v)
\]

Now, substituting for \( s \) its equivalent \( s'v \), we find

\[
p' = s'(v/C) = s'v/(c+v)
\]

which equation may also be expressed by the proportion

\[
p' : s' = v : C;
\]

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1 See [https://www.marxists.org/archive/marx/works/download/pdf/Capital-Volume-III.pdf](https://www.marxists.org/archive/marx/works/download/pdf/Capital-Volume-III.pdf)
the rate of profit is related to the rate of surplus-value as the variable capital is to the total capital. It follows from this proportion that the rate of profit, \( p' \), is always smaller than \( s' \), the rate of surplus-value, because \( v \), the variable capital, is always smaller than \( C \), the sum of \( v + c \), or the variable plus the constant capital; the only, practically impossible case excepted, in which \( v = C \), that is, no constant capital at all, no means of production, but only wages are advanced by the capitalist.” -Vol. 3, p. 33

Marx therefore derives the conclusion that the rate of profit is smaller than the rate of surplus value from the equation \( p' = s'/v/c+v \), which upon recognition that the rate of surplus value is the same as the rate of exploitation, matches up exactly with the Morishima-Seton equation, except for the fact that in our case the constant and variable capital pertain to the characteristic vector of output (which they clearly do not in Marx’s case). Marx also defines the rate of profit differently than we do. He defines it as the ratio of the surplus value to the total capital, whereas we define it in a way closer to how modern economists would - as the ratio of profit measured in dollars to the cost of production in dollars. These differences aside, Marx’s finding that the rate of profit is generally smaller than the rate of exploitation is basically the substance of the Morishima-Seton-Okishio Theorem. He points out the exception in which they’re equal, the “practically impossible” case where the constant capital is zero. Our Corollary 1 does not have this same exception because under the assumptions we have made, the constant capital has to be positive. Thus, the case where the rates of profit and exploitation are both positive and equal does not apply.

Although Marx defines the rate of profit differently than we do, it turns out that, if the constant and variable capitals \( C \) and \( V \) continue to be defined as those for the characteristic vector \( y \), then the two definitions of the rate of profit are equivalent, as we show in the next result.

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COROLLARY 2: THE RATE OF PROFIT IS EQUAL TO THE RATIO OF THE SURPLUS VALUE TO THE TOTAL CAPITAL (\( p' = S/(C+V) \)), WHERE \( S, C, \) AND \( V \) ARE THE SURPLUS VALUE, CONSTANT CAPITAL, AND VARIABLE CAPITAL, RESPECTIVELY, OF THE CHARACTERISTIC VECTOR.

Let \( S_i \) denote the surplus value per unit of good \( i \); that is, it’s the same \( S_i \) that appears in the decomposition of value formula \( \lambda_i = C_i + V_i + S_i \) (see equation (11) in Result 2). The total surplus value from the production of the amount of good \( i \) in the characteristic vector is then \( S_i y_i \), where \( y_i \) is the \( i \)th entry of the characteristic vector \( y \). We now define the surplus value \( S \) of the characteristic vector as the sum of the amounts \( S_i y_i \) across all goods \( i=1,\ldots,m \):

\[
S = \sum_{i=1}^{n} S_i y_i
\]  

(26)

This is consistent with how \( C \) and \( V \) are presently defined. Note that:

\[
C = \Lambda_1 A_1 y_1 + \Lambda_1 A_{11} y_{11}
\]  

(27a)
\[
\begin{align*}
&= \left[ \sum_{j=1}^{n} \lambda_j a_{j1} \ \sum_{j=1}^{n} \lambda_j a_{j2} \ \cdots \ \sum_{j=1}^{n} \lambda_j a_{jn} \right] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\
&+ \left[ \sum_{j=1}^{n} \lambda_j a_{jn+1} \ \sum_{j=1}^{n} \lambda_j a_{jn+2} \ \cdots \ \sum_{j=1}^{n} \lambda_j a_{jm} \right] \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ \vdots \\ y_m \end{bmatrix}
\end{align*}
\]

But since the constant capital per unit of good \(i\) is defined as \(C_i = \sum_{j=1}^{n} \lambda_j a_{ji}\) (see equation (10a) in Result 2), we can write (27b) as:

\[
C = \left[ \begin{array}{cccc}
C_1 & C_2 & \cdots & C_n \\
\cdot & \cdot & \cdots & \cdot
\end{array} \right] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} + \left[ \begin{array}{cccc}
C_{n+1} & C_{n+2} & \cdots & C_m \\
\cdot & \cdot & \cdots & \cdot
\end{array} \right] \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ \vdots \\ y_m \end{bmatrix}
\]

(28a)

\[
= \sum_{i=1}^{n} C_i y_i + \sum_{i=n+1}^{m} C_i y_i 
= \sum_{i=1}^{m} C_i y_i 
\]

(28b)

(28c)

so that our definition of the constant capital of the characteristic vector literally is the sum of the constant capitals of the amounts of each of the goods in the characteristic vector. The definition we have been using for \(V\), the variable capital of the characteristic vector, is (using the case of rigid demands; for the case of flexible demands just replace \(+\) with \(W\) in what follows):

\[
V = \omega \Lambda_{II} BL_I y_I + \omega \Lambda_{II} BL_{II} y_{II} 
= \left[ \begin{array}{cc}
\omega \Lambda_{II} BL_I & \omega \Lambda_{II} BL_{II}
\end{array} \right] \begin{bmatrix} y_I \\ y_{II} \end{bmatrix} 
= \omega \Lambda_{II} B \left[ \begin{array}{cccc}
L_I & L_{II} & \cdots & L_m \\
\cdot & \cdot & \cdots & \cdot
\end{array} \right] \begin{bmatrix} y_I \\ y_{II} \\ \vdots \\ y_m \end{bmatrix} 
= \omega \Lambda_{II} BL_1 \cdots \omega \Lambda_{II} BL_n \omega \Lambda_{II} BL_{n+1} \cdots \omega \Lambda_{II} BL_m \begin{bmatrix} y_1 \\ \vdots \\ y_n \\ y_{n+1} \\ \vdots \\ y_m \end{bmatrix}
\]

(29a)

(29b)

(29c)

(29d)

(29e)

Since the variable capital per unit of good \(i\) is defined in the case of rigid demands as \(V_i = \omega \Lambda_{II} BL_i\) (see equation (10b) in Result 2), equation (29e) can be written:

\[
V = \left[ \begin{array}{cccc}
V_1 & \cdots & V_n & V_{n+1} & \cdots & V_m
\end{array} \right] \begin{bmatrix} y_1 \\ \vdots \\ y_n \\ y_{n+1} \\ \vdots \\ y_m \end{bmatrix} 
= (V_1 y_1 + \cdots + V_n y_n) + (V_{n+1} y_{n+1} + \cdots + V_m y_m) 
= \sum_{i=1}^{n} V_i y_i
\]

(30a)

(30b)

(30c)
Thus our definition of the variable capital of the characteristic vector is literally the sum of the variable capitals of the amounts of each good in that vector.

Recall that (equation (10c) in Result 2) the surplus value per unit of good \(i\) (again for the case of rigid demands) is defined by:

\[ S_i = e \omega \Lambda_{ii} B l_i, \quad (31) \]

so that \( S_i = e V_i \) (see equation (12) in Result 2). Plugging this into (26) yields:

\[
S = \sum_{i=1}^{n} e V_i y_i \\
= e \sum_{i=1}^{n} V_i y_i \\
= e V
\]

Plugging this result into the Morishima-Seton equation (9) or (25) yields:

\[
\pi = \frac{e V}{C + V} \\
= \frac{S}{C + V}
\]

as was to be shown.

The definitions we have used for \(S, C,\) and \(V\) (the surplus value, constant capital, and variable capital of the characteristic vector \(y\)) are not the only conceivable definitions. We could also define them as the surplus value, constant capital, and variable capital of the actual vector of goods produced (call it \(x\)), and in some sense that would be a more useful definition since it would be more applicable to the situation at hand for the economy at any particular time. This raises the question of whether Marx’s rate of profit formula \(\pi = S/(C + V)\) holds with \(S, C,\) and \(V\) defined in terms of the vector of goods actually produced. We will explore this question more fully in a later document, but there is one easy case where it works, as shown in the next result.

**COROLLARY 3: IF THE RATE OF EXPLOITATION IS ZERO, THEN MARX’S RATE OF PROFIT FORMULA \(\pi = S/(C + V)\) HOLDS WITH \(S, C,\) AND \(V\) DEFINED AS THE SURPLUS VALUE, CONSTANT CAPITAL, AND VARIABLE CAPITAL, RESPECTIVELY, OF THE VECTOR OF GOODS ACTUALLY PRODUCED.**

Let \(x\) denote the vector of goods actually produced in the economy, with \(x_i\) the \(i^{th}\) entry of \(x\) (i.e., \(x_i\) is the amount of good \(i\) actually produced). With \(S, C,\) and \(V\) defined as the surplus value, constant capital, and variable capital, respectively, of the goods actually produced, we would now have:

\[
C = \sum_{j=1}^{m} C_i x_i
\]
\[ \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_m \end{bmatrix} \quad \text{where we have used } C_i = \sum_{j=1}^n \lambda_j a_{ji} \text{ for each } i. \]

Recall that \( \Lambda_I > 0 \) under Assumption Blocks 1-3 and every column of \( A_I \) and of \( A_{II} \) has a strictly positive entry under Assumption Block 2. Thus \( \Lambda_I A_I \) and \( \Lambda_I A_{II} \) are both strictly positive vectors. So, as long as the vector \( x \) of goods produced contains a positive amount of at least some good (which presumably it would or the label “goods produced” seems dubious), equation (34f) implies that \( C > 0 \).

Similarly, the variable capital of the goods produced would be defined (in the case of rigid demands) as:

\[ V = \sum_{j=1}^m V_j x_i \]

\[ \begin{bmatrix} V_1 & \cdots & V_n & V_{n+1} & \cdots & V_m \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_m \end{bmatrix} \]

\[ = [\omega \Lambda_{II} B l_1 \cdots \omega \Lambda_{II} B l_n \ \omega \Lambda_{II} B l_{n+1} \cdots \omega \Lambda_{II} B l_m] x \]

\[ = [\omega \Lambda_{II} B [l_1 \cdots l_n \ l_{n+1} \cdots l_m] x \]

\[ = [\omega \Lambda_{II} B l_l \ \omega \Lambda_{II} B l_{II}] x, \]

where we have used \( V_i = \omega \Lambda_{II} B l_i \) for each \( i \). The surplus value of the goods produced would be:

\[ S = \sum_{j=1}^m S_j x_i \]

\[ \begin{bmatrix} S_1 & \cdots & S_n & S_{n+1} & \cdots & S_m \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_m \end{bmatrix} \]

\[ = [e \omega \Lambda_{II} B l_1 \cdots e \omega \Lambda_{II} B l_n \ e \omega \Lambda_{II} B l_{n+1} \cdots e \omega \Lambda_{II} B l_m] x \]

\[ = [e \omega \Lambda_{II} B [l_1 \cdots l_n \ l_{n+1} \cdots l_m] x \]

\[ = [e \omega \Lambda_{II} B l_l \ e \omega \Lambda_{II} B l_{II}] x, \]

where we have used \( S_i = e \omega \Lambda_{II} B l_i \) for each \( i \). Now if the rate of exploitation \( e \) is zero, then the rate of profit \( \pi \) is also zero by Corollary 1. Meanwhile, (36f) implies \( S = 0 \). But since \( C > 0 \), we have \( C + V > 0 \), so \( S/(C + V) = 0 \), which is equal to \( \pi \). The formula works, as claimed.