RESULT 9: UNDER ASSUMPTION BLOCKS 1-4, IF ALL INDUSTRIES EARN POSITIVE PROFITS, THE WAGE-PRICE OF EACH COMMODITY IS GREATER THAN ITS LABOR VALUE.

Recall from the Fundamental Marxian Theorem that, if all industries are simultaneously earning positive profits, the following two vector inequalities hold (see (2c) and (3c) in that document):

\[ p_I > p_I A_I + w L_I \]  \hspace{1cm} (1a)
\[ p_{II} > p_I A_{II} + w L_{II} \]  \hspace{1cm} (1b)

where \( p_I \) is the vector of prices of capital goods, \( p_{II} \) is the vector of prices of wage/luxury goods, \( A_I \) is the capital input coefficient matrix for capital goods, \( A_{II} \) is the capital input coefficient matrix for wage/luxury goods, \( L_I \) is the labor input coefficient vector for capital goods, and \( L_{II} \) is the labor input coefficient vector for wage/luxury goods, and the wage \( w \) is equal to the cost of the hourly subsistence bundle of goods: \( w = \omega p_{II} B \), where \( \omega = 1/T \) is the fraction of the workday \( T \) taken up by an hour.

Dividing (1a) through by the wage (which is \( > 0 \) since \( \omega > 0 \), \( p_{II} > 0 \) by (1b), and \( B \) has at least one strictly positive entry by Assumption Block 4) gives:

\[ \frac{p_I}{w} > \frac{p_I}{w} A_I + L_I \]  \hspace{1cm} (2a)
\[ \Rightarrow p_{I,w} > p_{I,w} A_I + L_I \]  \hspace{1cm} (2b)
\[ \Rightarrow p_{I,w} - p_{I,w} A_I > L_I \]  \hspace{1cm} (2c)
\[ \Rightarrow p_{I,w} (I - A_I) > L_I \]  \hspace{1cm} (2d)

where \( p_{I,w} \) is the vector of wage-prices of capital goods (that is, each capital good price divided by the wage, which measures the price of the capital good denominated in units of labor instead of dollars) and \( L_I \) is the \((n \times n)\) identity matrix. Since \( p_{I,w} \) is a \((1 \times n)\) vector and \((I - A_I)\) is an \((n \times n)\) matrix, the product \( p_{I,w} (I - A_I) \) is \((1 \times n)\). Let its entries be denoted by \( y_1, y_2, ..., y_n \), so that:

\[ p_{I,w} (I - A_I) = [y_1 \ y_2 \ \cdots \ y_n] \]  \hspace{1cm} (3a)

Since the entries of \( L_I \) are \( l_1, l_2, ..., l_n \), (2d) can be written:

\[ [y_1 \ y_2 \ \cdots \ y_n] > [l_1 \ l_2 \ \cdots \ l_n], \]  \hspace{1cm} (4)

which means that \( y_i > l_i \) for \( i = 1, ..., n \). Now recall from Corollary 2 to Result 2 in “Morishima’s Labor Theory of Value Results” that, since \( A_I \) is productive under Assumption Block 3, the inverse of the net capital output matrix \((I - A_I)\) is a non-negative matrix in which each diagonal element is \( \geq 1 \). Letting the \((i,j)\) entry of \((I - A_I)^{-1}\) be denoted \( u_{ij} \), we can say that \( u_{ii} \geq 1 \) for \( i = 1, ..., n \) and \( u_{ij} \geq 0 \) for \( i \neq j \). Observe that since \( y_i > l_i \geq 0 \) and \( u_{ii} > 0 \) for any \( i \) between 1 and \( n \), we have:

\[ y_i u_{ii} > l_i u_{ii} \]  \hspace{1cm} (5)
Furthermore, for any \( j \neq i \) we have \( y_j > l_j \geq 0 \) and \( u_{ji} \geq 0 \), so:

\[ y_j u_{ji} \geq l_j u_{ji} \quad (6) \]

It follows from (5) and (6) that for every \( i=1,\ldots,n \) we have:

\[ y_1 u_{1i} + \cdots + y_i u_{ii} + \cdots + y_n u_{ni} > l_1 u_{1i} + \cdots + l_i u_{ii} + \cdots + l_n u_{ni} \]

\[ \Rightarrow [y_1 \ldots y_i \ldots y_n] \begin{bmatrix} u_{1i} \\ \vdots \\ u_{ii} \\ \vdots \\ u_{ni} \end{bmatrix} > [l_1 \ldots l_i \ldots l_n] \begin{bmatrix} u_{1i} \\ \vdots \\ u_{ii} \\ \vdots \\ u_{ni} \end{bmatrix} \quad (7a) \]

\[ \Rightarrow [p_{l,w}(I - A_i)] \cdot \text{col}_i(I - A_i)^{-1} > L_i \cdot \text{col}_i(I - A_i)^{-1} \]

If we stack all the inequalities in (7c) side-by-side into a vector inequality, we get:

\[ [p_{l,w}(I - A_i)] \cdot \text{col}_1(I - A_i)^{-1} \cdots p_{l,w}(I - A_i) \cdot \text{col}_n(I - A_i)^{-1}] > [L_1 \cdot \text{col}_1(I - A_i)^{-1} \cdots L_i \cdot \text{col}_n(I - A_i)^{-1}] \quad (8a) \]

\[ \Rightarrow p_{l,w}(I - A_i)[\text{col}_1(I - A_i)^{-1} \cdots \text{col}_n(I - A_i)^{-1}] > L_i[\text{col}_1(I - A_i)^{-1} \cdots \text{col}_n(I - A_i)^{-1}] \quad (8b) \]

\[ \Rightarrow p_{l,w}(I - A_i)(I - A_i)^{-1} > L_i(I - A_i)^{-1} \quad (8c) \]

\[ \Rightarrow p_{l,w} > L_i(I - A_i)^{-1} \quad (8d) \]

But recall that the labor value equation for capital goods is (see equation (1a) in Result 1 from “Morishima’s Labor Theory of Value Results”):

\[ \Lambda_1 = \Lambda_i A_i + L_i \quad (9a) \]

\[ \Rightarrow \Lambda_1 - \Lambda_i A_i = L_i \quad (9b) \]

\[ \Rightarrow \Lambda_i(I - A_i) = L_i \quad (9c) \]

\[ \Rightarrow \Lambda_i = L_i(I - A_i)^{-1} \quad (9d) \]

Substituting (9d) into (8d) we see that:

\[ p_{l,w} > \Lambda_i, \quad (10) \]

which says that the vector of wage prices of capital goods is strictly greater than the vector of labor values of capital goods. Thus we have proved the result in the case of capital goods. To prove it for wage and luxury goods, divide (1b) through by \( \Lambda \) to obtain:

\[ \frac{p_{II}}{w} > \frac{p_{l}}{w} A_{II} + L_{II} \]

\[ \Rightarrow p_{II,w} > p_{l,w} A_{II} + L_{II}, \quad (11b) \]
where \( p_{II,w} \) is the vector of wage-prices of wage and luxury goods (the prices of these goods denominated in units of labor rather than dollars). Note that since \( p_{I,w} \) is \((1 x n)\) and \( A_{II} \) is \((n x (m-n))\), the product \( p_{I,w} A_{II} \) is \((1 x (m-n))\). The \( j^{th} \) entry of this vector is:

\[
p_{I,w} \cdot \text{col}_j A_{II} = \begin{bmatrix} p_1 \ n \ p_2 \ n \ \ldots \ p_n \ n \end{bmatrix} \begin{bmatrix} a_{1,n+j} \\ a_{2,n+j} \\ \vdots \\ a_{n,n+j} \end{bmatrix}
\]

\[
= \frac{p_1}{w} a_{1,n+j} + \frac{p_2}{w} a_{2,n+j} + \ldots + \frac{p_n}{w} a_{n,n+j}
\]

(12a)

(12b)

Now since every column of \( A_{II} \) has at least one strictly positive entry under Assumption Block 2, there must be an \( i \) between 1 and \( n \) such that \( a_{i,n+j} > 0 \). Since \( p_i / w > \lambda_i \) according to (10), we have:

\[
\frac{p_i}{w} a_{i,n+j} > \lambda_i a_{i,n+j}
\]

(13)

For every \( k \neq i \), we at least know that \( a_{k,n+j} \geq 0 \) as \( A_{II} \) is a non-negative matrix by Assumption Block 1. Therefore, since \( p_k / w > \lambda_k \) according to (10), it follows that:

\[
\frac{p_k}{w} a_{k,n+j} \geq \lambda_k a_{k,n+j}
\]

(14)

Since the choice of \( j \) was arbitrary, (12b), (13) and (14) imply that for any \( j=1,\ldots,m-n \) we have:

\[
p_{I,w} \cdot \text{col}_j A_{II} = \frac{p_1}{w} a_{1,n+j} + \ldots + \frac{p_i}{w} a_{i,n+j} + \ldots + \frac{p_n}{w} a_{n,n+j}
\]

\[
> \lambda_1 a_{1,n+j} + \ldots + \lambda_i a_{i,n+j} + \ldots + \lambda_n a_{n,n+j}
\]

(15a)

(15b)

\[
= \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_n \end{bmatrix} \begin{bmatrix} a_{1,n+j} \\ a_{2,n+j} \\ \vdots \\ a_{n,n+j} \end{bmatrix}
\]

(15c)

\[
= \Lambda_i \text{col}_j A_{II}
\]

(15d)

Stacking the inequalities in (15d) side-by-side for \( j=1,\ldots,m-n \) into a vector inequality gives:

\[
[p_{I,w} \cdot \text{col}_1 A_{II} \ \ p_{I,w} \cdot \text{col}_2 A_{II} \ \ldots \ \ p_{I,w} \cdot \text{col}_{m-n} A_{II}] > [\Lambda_1 \text{col}_1 A_{II} \ \Lambda_1 \text{col}_2 A_{II} \ \ldots \ \Lambda_1 \text{col}_{m-n} A_{II}]
\]

(16a)

\[
\Rightarrow p_{I,w} [\text{col}_1 A_{II} \ \text{col}_2 A_{II} \ \ldots \ \text{col}_{m-n} A_{II}] > \Lambda_1 [\text{col}_1 A_{II} \ \text{col}_2 A_{II} \ \ldots \ \text{col}_{m-n} A_{II}]
\]

(16b)

\[
\Rightarrow p_{I,w} A_{II} > \Lambda_1 A_{II}
\]

(16c)

It then follows from (11b) that:

\[
p_{II,w} > \Lambda_1 A_{II} + L_{II}
\]

\[
= \Lambda_{II},
\]

(17a)

(17b)
where (17b) holds by the labor-value equation for wage and luxury goods (see equation (1b) in Result 1 of “Morishima’s Labor Theory of Value Results”). This shows that the vector of wage-prices for wage/luxury goods is strictly greater than the vector of labor values for these goods, so the result is now proved.

DISCUSSION

This result seems very intuitive. In order to earn a positive profit, the firm has to sell the product for more labor units (the wage-price) than it takes to directly and indirectly produce it (the labor value). Despite the intuitive appeal, it appears that Marx was unaware of this result, for he talked in a number of places about how some prices would be above labor values, while others would be below, as we see in these passages from Volume III:

“Taken together, the commodities are sold at $2 + 7 + 17 = 26$ above, and $8 + 18 = 26$ below their value, so that the deviations of price from value balance out one another through the uniform distribution of surplus-value, or through addition of the average profit of 22 per 100 units of advanced capital to the respective cost-prices of the commodities I to V. One portion of the commodities is sold above its value in the same proportion in which the other is sold below it.” -Vol. 3, p. 120

“The value of the commodities produced by capital II would, therefore, be smaller than their price of production, the price of production of the commodities of III smaller than their value, and only in the case of capital I in branches of production in which the composition happens to coincide with the social average, would value and price of production be equal.” -Vol. 3, p. 125

Note that we obtained a similar result to the last sentence in Result 7, where we showed that if the value composition of capital is the same for all goods, then prices are proportional to labor values. We did not, however, find that the prices are equal to labor values as Marx asserts.

Marx believed that the sum of the deviations of prices around labor values would be zero, so that in the aggregate the proposition that price = labor value would be true:

“The aggregate price of the commodities I to V would therefore equal their aggregate value, i. e., the sum of the cost-prices I to V plus the sum of the surplus-values, or profits, produced in I to V. It would hence actually be the money-expression of the total quantity of past and newly applied labour incorporated in commodities I to V. And in the same way the sum of the prices of production of all commodities produced in society – the totality of all branches of production – is equal to the sum of their values.” -Vol. 3, p. 122

“As for the variable capital, the average daily wage is indeed always equal to the value produced in the number of hours the labourer must work to produce the necessities of life. But this number of hours is in its turn obscured by the deviation of the prices of production of the necessities of life from their values. However, this always resolves itself to one commodity receiving too little of the surplus-value while another receives too much, so that the deviations from the value which are embodied in the prices of production compensate one another.” -Vol. 3, p. 123

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1 Quotes from Volume III of *Capital* from: https://www.marxists.org/archive/marx/works/download/pdf/Capital-Volume-III.pdf
Our result indicates that Marx’s conjecture is not correct, at least for the market prices of goods. Since each of the prices (converted into labor units) is higher than the corresponding labor value, the sum of all the wage-prices would have to be strictly larger than the sum of the labor values, not equal to them. According to Morishima, the problem seems to be that Marx occasionally conflated the price accounting system with the value accounting system. If one thinks about the units of measurement for prices and values, they can’t actually be directly comparable since the former is in dollars and the latter in labor hours. So, in order to be compared they have to be converted into the same units, as we have done here in dividing all prices by the wage. (Indeed, Marx seems to recognize the need to convert them into the same units in the above quote from p. 122). But in doing so we find that all the wage-prices are greater than the corresponding labor values, rather than some being above value and some being below. It turns out that Marx was correct about a certain type of prices, which Morishima calls the “Marxian prices” and which are measured in labor units, but not the actual market prices of goods which are measured in dollars. To delve more fully into this issue, we must turn to a discussion of the “transformation problem,” which concerns how to convert the labor values of goods into their prices of production and how to convert between surplus values and profits.