RESULT 8: UNDER ASSUMPTION BLOCKS 1-5, IF ALL INDUSTRIES HAVE THE SAME VALUE COMPOSITION OF CAPITAL, THEN MARX’S RATE OF PROFIT FORMULA \((\pi = S(C+V))\) HOLDS, WHERE \(S, C,\) AND \(V\) ARE THE SURPLUS VALUE, CONSTANT CAPITAL AND VARIABLE CAPITAL, RESPECTIVELY, OF THE VECTOR OF GOODS ACTUALLY PRODUCED.

As shown in Existence of Capitalist Equilibrium, under Assumption Blocks 1-5 we can identify a unique rate of profit \(\pi \geq 0\) and price vector \(p > 0\) that satisfy the price-determining equations:

\[
p = (1 + \pi)pM, \tag{1}
\]

where \(M\) is the input power matrix and the wage \(w\) is equal to the cost of the hourly subsistence bundle of goods. As also explained in Existence of Capitalist Equilibrium, the respective price determining equations for capital goods and for wage/luxury goods are:

\[
\begin{align*}
p_i &= (1 + \pi)(p_i A_i + p_{II} \omega BL_i) \quad \text{for } i=1,\ldots,n \tag{2a} \\
p_{II} &= (1 + \pi)(p_{II} A_{II} + p_{II} \omega BL_{II}) \quad \text{for } i=n+1,\ldots,m \tag{2b}
\end{align*}
\]

where \(p_i\) is the vector of prices of capital goods, \(p_{II}\) is the vector of prices of wage/luxury goods, \(A_i\) is the capital input matrix for capital goods, \(A_{II}\) is the capital input matrix for wage and luxury goods, \(L_i\) is the labor input vector for capital goods, \(L_{II}\) is the labor input vector for wage/luxury goods, \(\omega\) is the fraction of the workday \(T\) taken up by an hour, and \(B\) is the vector of daily subsistence amounts of wage and luxury goods (so that \(\omega B\) is the hourly subsistence bundle).

Writing out the \(i\)th entry of each of the vector equations (2a) and (2b), we obtain:

\[
\begin{align*}
p_i &= (1 + \pi)(\sum_{j=1}^{n} p_j a_{ji} + p_{II} \omega B l_i) \quad \text{for } i=1,\ldots,n \tag{3a} \\
p_i &= (1 + \pi)(\sum_{j=1}^{n} p_j a_{ji} + p_{II} \omega B l_i), \quad \text{for } i=n+1,\ldots,m \tag{3b}
\end{align*}
\]

where \(p_i\) is the price of good \(i\), \(a_{ji}\) is the number of units of capital good \(j\) required for the production of a unit of good \(i\), and \(l_i\) is the number of units of labor required for the production of a unit of good \(i\). We can condense (3a) and (3b) into a single equation:

\[
p_i = (1 + \pi)(\sum_{j=1}^{n} p_j a_{ji} + p_{II} \omega B l_i) \quad \text{for } i=1,\ldots,m \tag{4}
\]

Now assume that all industries have the same value composition of capital and let \(k\) denote this common value, so that \(C_i/V_i = k\) for \(i=1,\ldots,m\), where \(C_i = \sum_{j=1}^{n} \lambda_j a_{ji}\) is the constant capital of good \(i\) (\(\lambda_j\) being the value of good \(j\)) and \(V_i = \omega \Lambda_{II} B l_i\) is the variable capital of good \(i\) (\(\Lambda_{II}\) being the vector of labor values of wage and luxury goods). Note that \(C_i = \Lambda_i c_1 l_1 A_1\) for \(i=1,\ldots,n\) and \(C_i = \Lambda_i c_{m-1} l_{m-1} A_{II}\) for \(i=n+1,\ldots,m\), where \(\Lambda_i\) is the vector of labor values of capital goods. Recall that \(\Lambda_i > 0\) under Assumption Blocks 1-3, and each column of \(A_i\) has at least one strictly positive entry (and the same for \(A_{II}\)) under Assumption Block 2. Thus we have \(C_i > 0\) for all \(i\). Note also that \(\omega = T/3 > 0\), that \(\Lambda_{II} > 0\) under Assumption Blocks 1-3, that \(B\) has at least one strictly positive entry by Assumption Block 4, and that \(l_i > 0\) for at least some \(i\) by Assumption Block 2. Therefore, \(V_i > 0\) for at least some \(i\), which means that \(C_i/V_i > 0\) and is finite for that \(i\). But then \(C_i/V_i > 0\) and is finite for every \(i\), since all of these values are supposed to be the same.
Thus $k$ is a positive, finite real number. Also note that if $C_i/V_i > 0$ and is finite with $C_i > 0$ for every $i$, then $V_i > 0$ must hold for every $i$; otherwise, if $V_i$ were 0 for some $i$, then $C_i/V_i$ would be infinite.

By Result 7, the fact that all goods have the same (finite) value composition of capital implies that prices are proportional to labor values, so there exists a number $\alpha > 0$ (recall that prices are strictly positive by Existence of Capitalist Equilibrium and labor values are strictly positive under Assumption Blocks 1-3) such that:

$$p = \alpha \Lambda,$$  \hspace{1cm} (5)

where $\Lambda$ is the vector of labor values of all goods. The meaning of (5) is that $p_i = \alpha \lambda_i$ for every good $i = 1, \ldots, m$; that is, the ratio of price to value is the same for all goods. Plugging this into (4) gives:

$$\alpha \lambda_i = (1 + \pi)(\sum_{j=1}^n \alpha \lambda_j a_{ji} + p_{II} \omega B_l)$$  \hspace{1cm} for $i = 1, \ldots, m$  \hspace{1cm} (6)

But note that $p_i = \alpha \lambda_i$ for all $i$ means that $p_i = \alpha \Lambda_i$ and $p_{II} = \alpha \Lambda_{II}$. Using $p_{II} = \alpha \Lambda_{II}$ in (6) we find:

$$\alpha \lambda_i = (1 + \pi)(\sum_{j=1}^n \alpha \lambda_j a_{ji} + \alpha \Lambda_{II} \omega B_l)$$  \hspace{1cm} (7a)

$$\Rightarrow \alpha \lambda_i = (1 + \pi) \alpha(\sum_{j=1}^n \lambda_j a_{ji} + \Lambda_{II} \omega B_l)$$  \hspace{1cm} (7b)

$$\Rightarrow \lambda_i = (1 + \pi)(\sum_{j=1}^n \lambda_j a_{ji} + \omega \Lambda_{II} B_l)$$  \hspace{1cm} (7c)

$$\Rightarrow \lambda_i = (1 + \pi)(C_i + V_i)$$  \hspace{1cm} (7d)

for $i = 1, \ldots, m$, where we have used $\alpha > 0$ in going from (7b) to (7c) and the definitions of $C_i$ and $V_i$ in moving from (7c) to (7d). Solving (7d) for the rate of profit, we obtain:

$$\lambda_i = (C_i + V_i) + \pi(C_i + V_i)$$  \hspace{1cm} (8a)

$$\Rightarrow \pi(C_i + V_i) = \lambda_i - C_i - V_i$$  \hspace{1cm} (8b)

$$\Rightarrow \pi(C_i + V_i) = S_i$$  \hspace{1cm} (8c)

$$\Rightarrow \pi = \frac{S_i}{C_i + V_i}$$  \hspace{1cm} (8d)

for $i = 1, \ldots, m$, where we have used the decomposition of value $\lambda_i = C_i + V_i + S_i$ to move from (8b) to (8c) and the fact that $C_i + V_i > 0$ to move from (8c) to (8d). Using $S_i = eV_i$ and $C_i = kV_i$ for all $i$, (8d) becomes:

$$\pi = \frac{eV_i}{kV_i + V_i}$$  \hspace{1cm} (9a)

$$\Rightarrow \pi = \frac{eV_i}{(k+1)V_i}$$  \hspace{1cm} (9b)

$$\Rightarrow \pi = \frac{e}{(k+1)}.$$  \hspace{1cm} (9c)
where we have used \( V_i > 0 \) for all \( i \) to get from (9b) to (9c). But if \( S, C, \) and \( V \) are defined as the surplus value, constant capital, and variable capital, respectively, of the vector of goods actually produced (denote it \( x \)), then we have:

\[
V = \sum_{i=1}^{m} V_i x_i \tag{10}
\]

\[
S = \sum_{i=1}^{m} S_i x_i = \pi \sum_{i=1}^{m} e V_i x_i = e \sum_{i=1}^{m} V_i x_i = eV \tag{11a}
\]

\[
C = \sum_{i=1}^{m} C_i x_i = \sum_{i=1}^{m} k V_i x_i = k \sum_{i=1}^{m} V_i x_i = kV \tag{11b}
\]

where we have used (10) in deriving (11d) and (12d). Note that \( V > 0 \) since \( V_i > 0 \) for all \( i \) and the vector of goods produced \( x \) is a non-negative, nonzero vector (if it were the zero vector then it would not warrant the label of goods “produced”). It follows from (11d), (12d), and (9c) that:

\[
\frac{S}{C+V} = \frac{eV}{kV+V} = \frac{e}{k+1} \frac{V}{V} = \frac{e}{k+1} \{\text{since } V > 0\} \tag{13a}
\]

\[
\frac{S}{C+V} = \frac{e}{k+1} \pi \tag{13b}
\]

so that Marx’s rate of profit formula holds, as claimed.

The next result gives another, even more restrictive case in which Marx’s formula is valid.

**COROLLARY:** IF ALL INDUSTRIES HAVE THE SAME INTERNAL COMPOSITION OF CAPITAL, THEN MARX’S RATE OF PROFIT FORMULA \( (\pi = S/(C+V)) \) HOLDS, WITH \( S, C, \) AND \( V \) THE SURPLUS VALUE, CONSTANT CAPITAL AND VARIABLE CAPITAL, RESPECTIVELY, OF THE VECTOR OF GOODS ACTUALLY PRODUCED

Assume that all industries have the same internal composition of capital.¹ This means every column of the input power matrix \( M \) is proportional to every other column. Thus \( M \) is a singular matrix in a very major way; it doesn’t have just one column that linearly depends on some subset of the other columns, but instead every column depends linearly on each of the other columns. Since the \( j \)th column of \( M \) gives the capital input coefficients \( a_{kj} \) for \( k=1,\ldots,n \) and labor power coefficients \( \omega b_k l_j \) for \( k=n+1,\ldots,m \) for good \( j \), saying that the \( j \)th column is a constant multiple of

¹ “Internal composition of capital” is an idea that Morishima attributes to the famous economist and Nobel Laureate Paul Samuelson. The first American to win the Nobel Prize in Economics, Samuelson was not a friend of Marxism. Consider the titles of a couple of his articles: “…A Modern Dissection of Marxian Economic Models” and “…A Summary of the So-Called Transformation Problem”. Yet it should be noted that, unlike so many others, he at least took the time to apply his (prodigious) skills to a careful study of the subject.
some other column $i$ means that there exists a number $\eta_{ji} > 0$ such you can multiply all the capital and labor power coefficients for good $j$ by $\eta_{ji}$ and obtain those coefficients for good $i$. To be precise, we have $a_{kj} = \eta_{ji}a_{ki}$ for all $k=1,\ldots,n$ and $\omega b_{kj} = \eta_{ji}\omega b_{ki}$ for all $k=n+1,\ldots,m$. Thus, the production process for producing good $i$ is just a scaled up or scaled down version of the process for producing good $j$. In this sense the production processes are all redundant with respect to each other; they have an “if you’ve seen one, you’ve seen them all” type of character. Obviously this is a highly restrictive assumption that would seldom if ever play out in real life, but we nevertheless proceed to show that it implies the validity of Marx’s rate of profit formula.²

We will show that if all industries have the same internal composition of capital, they must have the same value composition as well. Then the corollary will follow immediately from Result 8. Observe that the constant capital of industry $j$ is:

$$C_j = \sum_{k=1}^{n} \lambda_k a_{kj}$$

(14a)

$$= \sum_{k=1}^{n} \lambda_k \eta_{ji} a_{ki}$$

(14b)

$$= \eta_{ji} \sum_{k=1}^{n} \lambda_k a_{ki}$$

(14c)

$$= \eta_{ji} C_i,$$  

(14d)

where $i$ is any industry other than $j$. Meanwhile, the variable capital of industry $j$ is:

$$V_j = \omega \Lambda_{1j} B l_j$$

(15a)

$$= \omega (\sum_{k=n+1}^{m} \lambda_k b_{kj}) l_j$$

(15b)

$$= \sum_{k=n+1}^{m} \lambda_k (\omega b_{kj} l_j)$$

(15c)

$$= \sum_{k=n+1}^{m} \lambda_k (\eta_{ji} \omega b_{ki} l_i)$$

(15d)

$$= \eta_{ji} \omega (\sum_{k=n+1}^{m} \lambda_k b_{kj}) l_i$$

(15e)

$$= \eta_{ji} \omega \Lambda_{1j} B l_i$$

(15f)

$$= \eta_{ji} V_i,$$  

(15g)

where again $i$ is any industry other than $j$. Using (14d) and (15g), the value composition of capital for industry $j$ is:

$$\frac{c_j}{v_j} = \frac{\eta_{ji} c_i}{\eta_{ji} v_i}$$  

(16a)

$$= \frac{c_i}{v_i},$$  

(16b)

where $i$ is any industry other than $j$ and we have used $\eta_{ji} > 0$ to get from (16a) to (16b). Thus the value compositions of industries $i$ and $j$ are the same. Since $i$ and $j$ were chosen arbitrarily, we can conclude that all industries have the same value composition of capital. By Result 8, Marx’s rate of profit formula holds, and the proof is complete.

² A common criticism of Marxian economics over the years seems to be that it requires very restrictive assumptions like equal value composition or internal composition of capital to show that prices and labor values are proportional. But this view assumes that Marx’s objective was to argue that they should be proportional. As Morishima points out, this was not his objective. Rather, his aim was to show that capitalism tends to obscure this relationship, unless very restrictive assumptions hold. In that sense, he was successful, within the confines of the labor theory of value.