RESULT 4:  
IF CAPITALIST SOCIETY IS VIABLE, THEN THE CAPITAL INPUT MATRIX IS PRODUCTIVE.

In order to prove this result we have to clarify what it means for capitalist society to be viable. It means that all industries in the economy need to be able to simultaneously earn positive profits. Any industry that does not accomplish this will cease to exist. If this happened for any wage goods, it would be catastrophic for society, as these goods are by definition necessary for survival. If it happened for any capital goods, it would likewise cause a system crash because we assume that every capital good is essential for the production of at least some other good. The only industries that could shut down without causing the system to implode would be luxury goods, since they are not fundamental to survival. However, for any such industries we would simply remove them from the list of society’s industries. The ones that remain among the list of viable luxury good industries would then have to be earning positive profits.

We now need to develop precise mathematical conditions that correspond to all industries simultaneously earning strictly positive profits. Recall that profit for any firm is equal to total revenue minus total cost of production. So consider the industry that produces capital good 1. If we let $p_1$ be the price and $x_1$ the total industry output of this good, total industry revenue is $p_1 x_1$. If we let $p_2, p_3, \ldots, p_n$ be the prices of capital goods 2, 3, \ldots, $n$, then since it takes $a_{11}, a_{21}, \ldots, a_{n1}$ units of capital goods 1 through $n$ to produce each unit of capital good 1, the total capital cost of producing each unit of the good is $(p_1 a_{11} + p_2 a_{21} + \cdots + p_n a_{n1})$. Thus the total capital cost of producing the industry output of $x_1$ units $(p_1 a_{11} + p_2 a_{21} + \cdots + p_n a_{n1}) x_1$. Since it takes $l_1$ units of labor to produce each unit of output, the amount of labor used to produce $x_1$ units is $l_1 x_1$. If the wage paid for each unit of labor is $w$, then the total labor cost of producing $x_1$ units of output is $w l_1 x_1$. Thus the total cost of production is $(p_1 a_{11} + p_2 a_{21} + \cdots + p_n a_{n1}) x_1 + w l_1 x_1$ and profit is:

$$\pi_1 = p_1 x_1 - (p_1 a_{11} + p_2 a_{21} + \cdots + p_n a_{n1}) x_1 - w l_1 x_1$$  \hspace{1cm} (1a)$$

$$\Rightarrow p_1 - [p_1 \  p_2 \ \cdots \ p_n] \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} - w l_1 > 0$$  \hspace{1cm} (2a)$$

Obviously profit cannot be strictly positive if $x_1 = 0$, so we can assume $x_1 > 0$. In that case $\pi_1$ is $> 0$ if and only if profit per unit is $> 0$, i.e.:

$$p_1 - (p_1 a_{11} + p_2 a_{21} + \cdots + p_n a_{n1}) - w l_1 > 0$$  \hspace{1cm} (2b)$$

$$\Leftrightarrow p_1 - p_l \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} - w l_1 > 0,$$  \hspace{1cm} (2c)$$

where $p_l$ is the vector of prices of capital goods. Similarly, for any capital good $i=1,2,\ldots,n$, profit for industry $i$ will be strictly positive if and only if:
\[ p_l - p_l \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix} - wl_i > 0 \quad (3) \]

Stacking all of the inequalities of the form (3) side-by-side for \( i = 1, 2, \ldots, n \) gives us the following vector inequality:

\[
\begin{bmatrix}
    p_1 - p_l \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} - wl_1 \\
p_2 - p_l \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} - wl_2 \\
\vdots \\
p_n - p_l \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} - wl_n
\end{bmatrix} > \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]

\( \iff \begin{bmatrix} p_1 & p_2 & \ldots & p_n \\
p_1 & \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} - w[l_1 & l_2 & \ldots & l_n] \end{bmatrix} > \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \]  \( (4a) \)

\[ \iff p_l - p_l A_l - wL_l > 0, \quad (4b) \]

\[ \iff p_l > p_l A_l + wL_l \quad (4c) \]

where \( A_l \) is the capital input coefficient matrix for capital goods, \( L_l \) is the labor input coefficient vector for capital goods, and the “0” on the right hand side of the inequality is the \((1 \times n)\) zero vector. Inequality \((4d)\) is regarded as the condition for the viability of all capital goods industries. Note that in order for \((4d)\) to hold, \( p_l \) must be a strictly positive vector because every element of the vector on the right must be at least 0.

To derive a similar condition for wage and luxury goods, consider an arbitrary wage or luxury good \( j = n + 1, n + 2, \ldots, m \). If \( p_j \) is the price of good \( j \), \( x_j \) is industry output of good \( j \), then industry profit is:

\[
\pi_j = p_j x_j - \left( p_1 a_{1j} + p_2 a_{2j} + \cdots + p_n a_{nj} \right) x_j - w_l j x_j
\]

\[ = \left[ p_j - \left( p_1 a_{1j} + p_2 a_{2j} + \cdots + p_n a_{nj} \right) - w_l j \right] x_j, \]

\[ \quad (5a) \]

where \( a_{1j}, a_{2j}, \ldots, a_{nj} \) are the amounts of capital goods 1 through \( n \) required to produce a unit of good \( j \), and \( l_j \) is the amount of labor required to produce a unit of \( j \). Since \( x_j > 0 \) must hold, industry profit is strictly positive if and only if:

\[ p_j - \left( p_1 a_{1j} + p_2 a_{2j} + \cdots + p_n a_{nj} \right) - w_l j > 0 \]

\[ \iff p_j - \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} - w_l j > 0 \]

\[ \iff p_j - [p_1 & p_2 & \cdots & p_n] - [a_{1j} & a_{2j} & \cdots & a_{nj}] - w_l j > 0 \quad (6a) \]
\( \Leftrightarrow p_j - p_l \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} - w l_j > 0 \)  

(6c)

Stacking these inequalities side by side for \( j=n+1, n+2, \ldots, m \), we get:

\[
\begin{bmatrix}
p_{n+1} - p_l \begin{bmatrix} a_{1,n+1} \\ a_{2,n+1} \\ \vdots \\ a_{n,n+1} \end{bmatrix} - w l_{n+1} 
p_{n+2} - p_l \begin{bmatrix} a_{1,n+2} \\ a_{2,n+2} \\ \vdots \\ a_{n,n+2} \end{bmatrix} - w l_{n+2} 
\vdots 
p_m - p_l \begin{bmatrix} a_{1,m} \\ a_{2,m} \\ \vdots \\ a_{n,m} \end{bmatrix} - w l_m
\end{bmatrix} > \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]

(7a)

\[
\Leftrightarrow \begin{bmatrix} p_{n+1} & p_{n+2} & \cdots & p_m \end{bmatrix} - p_l \begin{bmatrix} a_{1,n+1} & a_{1,n+2} & \cdots & a_{1,m} \\ a_{2,n+1} & a_{2,n+2} & \cdots & a_{2,m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,n+1} & a_{n,n+2} & \cdots & a_{n,m} \end{bmatrix} - w [l_{n+1} \ l_{n+2} \ \cdots \ l_m] > \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]

(7b)

\[ \Leftrightarrow p_{II} - p_l A_{II} - w L_{II} > 0 \]

(7c)

\[ \Leftrightarrow p_{II} > p_l A_{II} + w L_{II}, \]

(7d)

where \( p_{II} \) is the vector of prices of wage and luxury goods, \( A_{II} \) is the capital input coefficient matrix for wage and luxury goods, and \( L_{II} \) is the labor input coefficient vector for wage and luxury goods. Inequality (7d) is the condition for the viability of wage/luxury goods producing industries. In order for (7d) to hold, \( p_{II} \) must be > 0 because the vector on the right is at least 0.

To sum up, capitalist society is viable if and only if the following two conditions are met:

\[ p_l > p_l A_l + w L_l \]

(4d)

\[ p_{II} > p_l A_{II} + w L_{II}, \]

(7d)

with \( p_l > 0 \) and \( p_{II} > 0 \).

Assuming that these conditions hold, (4d) implies:

\[ p_l > p_l A_l, \]

(8)

since \( w \geq 0 \) and \( L_l \geq 0 \). Taking transposes of both sides of (8) gives:

\[ p'_l > A'_l p'_l, \]

(9)

with \( p'_l > 0 \) since \( p_l > 0 \). This shows that \( A'_l \) is productive. But then \( A_l \) is productive by Result 3. This completes the proof.
An interpretation of this result is that society has to be reasonably developed in its productive capabilities in order to maintain capitalism as an economic system. Specifically, it has to be far enough along that there exists some production plan for capital goods from which it can obtain positive net outputs of all capital goods. (Of course, by Result 2, this means that it can actually produce any vector of capital goods as a net output). If this condition is not met, capitalism will fail because not all industries will be able to simultaneously earn positive profits.

The idea that society needs to reach a certain stage of productive/industrial development in order to be able to feasible adopt capitalism appears in the writings of both Marx and Lenin:

“The appearance of products as commodities presupposes such a development of the social division of labour, that the separation of use-value from exchange-value, a separation which first begins with barter, must already have been completed… Yet we know by experience that a circulation of commodities relatively primitive, suffices for the production of all these forms. Otherwise with capital. The historical conditions of its existence are by no means given with the mere circulation of money and commodities. It can spring into life, only when the owner of the means of production and subsistence meets in the market with the free labourer selling his labour-power. And this one historical condition comprises a world’s history. Capital, therefore, announces from its first appearance a new epoch in the process of social production.”¹ (Marx, Capital, Volume I, p. 120)

“The basic process of the formation of a home market (i.e., of the development of commodity production and of capitalism) is the social division of labour. This consists of various forms of processing raw materials (and various operations in this processing) separating from agriculture one after another and becoming independent branches of industry, which exchange their products (now commodities) for the products of agriculture. Thus, agriculture itself becomes industry (i.e., produces commodities), and the same process of specialisation takes place in it.” (Lenin, The Development of Capitalism in Russia, Chapter 1, Section IX)²

“The separation of the direct producer from the means of production, i.e., his expropriation, signifying the transition from simple commodity production to capitalist production (and constituting the necessary condition for this transition), creates the home market. The process of this creation of the home market proceeds in two directions: on the one hand, the means of production from which the small producer is “freed” are converted into capital in the hands of their new owner, serve to produce commodities and, consequently, are themselves converted into commodities.” (Lenin, The Development of Capitalism in Russia, Chapter I, Section IX)²

“The home market appears when commodity economy appears; it is created by the development of this commodity economy, and the degree to which the social division of labour is ramified determines the level of its development; it spreads with the extension of commodity production from products to labour-power, and only in proportion as the latter is transformed into a commodity does capitalism embrace the entire production of the country, developing mainly on account of means of production, which occupy an increasingly important place in capitalist society. The ‘home market’ for capitalism is created by developing capitalism itself, which

² See https://www.marxists.org/archive/lenin/works/1899/dcr8i/i8ix.htm
deepens the social division of labour and resolves the direct producers into capitalists and workers.” (Lenin, *The Development of Capitalism in Russia*, Chapter I, Section IX)²

Even if society has developed to the point of having a productive capital input matrix, it doesn’t guarantee that capitalism will be viable. Productivity of the input matrix is a necessary condition, but not a sufficient one. This is because, while it does enable condition (4d) to hold (as long as the wage and/or labor input requirements are low enough— an interesting observation in its own right) it doesn’t guarantee that the condition (7d) for strictly positive profits in wage and luxury good industries will hold. This echoes the idea put forward by both Marx and Lenin that just because conditions were such that capitalism could take hold, does not mean it would:

“Merchant’s and usurer’s capital always historically precede the formation of industrial capital and are logically the necessary premise of its formation (Das Kapital, III, 1, S. 312-316; Russ. trans., pp. 262-265; III, 2, 132-137, 149; Russ. trans., pp. 488-492, 502); but in themselves neither merchant’s capital nor usurer’s capital represents a sufficient premise for the rise of industrial capital (i.e., capitalist production); they do not always break up the old mode of production and replace it by the capitalist mode of production; the formation of the latter ‘depends entirely upon the stage of historical development and the attendant circumstances’ (ibid., 2, 133; Russ. trans., p. 489).” -Lenin, *The Development of Capitalism in Russia*, Chapter II, Section XIII³

The key to the actual transition to capitalism, according to both Marx and Lenin, is the development of mechanized industrial manufacture:

“So then, the data on Russian factory workers fully confirm the theory of Capital that it is large-scale machine industry that brings about a complete and definite revolution in the conditions of life of the industrial population, separating it once and for all from agriculture and from the century-old traditions of patriarchal life connected with it.” -Lenin, *The Development of Capitalism in Russia*, Chapter VII, Section XI⁴

³ See https://www.marxists.org/archive/lenin/works/1899/dcr8ii/ii8xiii.htm
⁴ See https://www.marxists.org/archive/lenin/works/1899/dcr8vii/vii8xi.htm