RESULT 3:
THE CAPITAL INPUT MATRIX IS PRODUCTIVE IF AND ONLY IF ITS TRANSPOSE IS PRODUCTIVE

This is another result of the form $X$ if and only if $Y$, where in this case $X$ is the statement “the capital input matrix is productive” and $Y$ is the statement “the transpose of the capital input matrix is productive.” To prove it, we must show that $X \Rightarrow Y$ and $Y \Rightarrow X$.

The capital input matrix is productive $\Rightarrow$ the transpose of the capital input matrix is productive.

Suppose the capital input matrix $A_i$ is productive. For each $j=1,2,…,n$ let $i_j$ be the $n \times 1$ column vector with 1 in the $j^{th}$ entry and 0 in every other entry. Then since $A_i$ is productive, Result 2 implies that there must exist a vector $x_j \geq 0$ such that $(I - A_i)x_j = i_j$. Note that $x_j$ cannot be the zero vector, for otherwise we would have $0 = i_j$, so it must contain at least one strictly positive entry. It follows from $(I - A_i)x_j = i_j$ that:

$$x_j = A_i x_j + i_j \quad (1)$$

Let $g$ be a strictly positive $1 \times n$ row vector and consider the vector $y = g(I - A_i)^{-1}$. Note that the productivity of $A_i$ implies that $(I - A_i)^{-1}$ exists by the Corollary to Result 2. Then we have:

$$y(I - A_i) = g \quad (2a)$$
$$\Rightarrow y - yA_i = g \quad (2b)$$
$$\Rightarrow y = yA_i + g \quad (2c)$$

Now note that (1) and (2c) respectively imply:

$$yx_j = yA_i x_j + yi_j \quad (3a)$$
and

$$yx_j = yA_i x_j + gx_j, \quad (3b)$$

from which we can conclude that:

$$yi_j = gx_j \quad (4a)$$
$$\Rightarrow y_j = gx_j, \quad (4b)$$

where $y_j$ is the $j^{th}$ entry of the vector $y$. Since $g$ is a strictly positive vector and $x_j$ is non-negative with at least one strictly positive entry, we have $gx_j > 0$, or $y_j > 0$ by (4b). Since this is true for any $j=1,2,…,n$, we see that $y$ is a strictly positive vector. Furthermore, (2c) implies $y > yA_i$ since $g > 0$. Thus the transpose of $y$ has to be strictly greater than the transpose of $yA_i$, so we get:

$$y' > (yA_i)' \quad (5a)$$
$$\Rightarrow y' > A_i'y' \quad (5b)$$
$$\Rightarrow (I - A_i')y' > 0 \quad (5c)$$
Inequality (5b) follows from a result in linear algebra which says that the transpose of a product matrix is the product of the transposes in reverse order. Thus we have found a vector $y'$ (which is $> 0$ since $y > 0$) such that $(I - A_i')y' > 0$. This shows that $A_i'$ is productive, which is what we wanted to show.

*The transpose of the capital input matrix is productive $\Rightarrow$ The capital input matrix is productive*

Suppose that $A_i'$ is productive. Then by the above argument, the transpose of $A_i'$ is productive. But the transpose of the transpose is just the matrix itself. So we see that $A_i$ is productive, as we wanted to show.

Since we have established both arrows of implication, the proof is now complete.