RESULT 1:
THE TWO ALTERNATIVE DEFINITIONS OF LABOR VALUE ARE EQUIVALENT

In *Capital*, Marx gives two alternative definitions of the labor value of a good:

(i) the total amount of labor embodied in a unit of the good

“Let us now consider the residue of each of these products; it consists of the same unsubstantial reality in each, a mere congelation of homogeneous human labour, of labour power expended without regard to the mode of its expenditure. All that these things now tell us is, that human labour power has been expended in their production, that human labour is embodied in them. When looked at as crystals of this social substance, common to them all, they are – Values.”
(Capital, Volume I, p. 28)

(ii) the amount of labor socially necessary for producing a unit

“We see then that that which determines the magnitude of the value of any article is the amount of labour socially necessary, or the labour time socially necessary for its production. Each individual commodity, in this connexion, is to be considered as an average sample of its class. Commodities, therefore, in which equal quantities of labour are embodied, or which can be produced in the same time, have the same value. The value of one commodity is to the value of any other, as the labour time necessary for the production of the one is to that necessary for the production of the other. 'As values, all commodities are only definite masses of congealed labour time.'” (p. 29)

“We know that the value of each commodity is determined by the quantity of labour expended on and materialized in it, by the working-time necessary, under given social conditions, for its production.” (p. 132)


These definitions sound the same, and according to Morishima, Marx in fact regarded them as the same, but their mathematical expressions are distinct.

The first definition is the one we gave in the “Model setup” post because it refers to the amount of labor required to produce all the capital needed to produce the good plus the amount of labor required as a direct input. In that post, we saw that the labor value vectors for the capital goods and the wage/luxury goods satisfy the following respective labor value equations:

\[
\Lambda_I = \Lambda_I A_I + L_I
\]

\[
\Lambda_{II} = \Lambda_I A_{II} + L_{II}
\]

(1a)  (1b)

So these equations can be regarded as a mathematical formulation of Marx’s first definition.
The second definition is tougher to formulate in a precise way. Recall from the “Model setup" post that if \( x_l = (x_1 \ x_2 \ \cdots \ x_n)' \) is a vector of production amounts for the capital goods,\(^1\) i.e., we’re going to produce \( x_1 \) units of capital good 1, \( x_2 \) units of capital good 2, \ldots, and \( x_n \) units of capital good \( n \), then the vector of net outputs of capital goods resulting from this production plan is:

\[
(I - A_l)x_l, \tag{2}
\]

where \( I \) represents the \( n \times n \) identity matrix. For the sake of what follows, let \( i_j \) denote the \( (n \times 1) \) column vector that has 1 in the \( j^{th} \) entry and 0 in all other entries. Now suppose we find a production vector of capital goods, call it \( x_l^{(1)} \), that yields a net output of one unit of capital good 1 and zero units of all other capital goods. Then it follows that \( (I - A_l)x_l^{(1)} = i_1 \). Does such a production vector necessarily exist? It turns out that, as we will see later, under certain conditions on \( A_l \), it does have to exist. Similarly, find a production vector \( x_l^{(2)} \) that yields one unit of good 2 and zero units of all other goods as net outputs. Then we have \( (I - A_l)x_l^{(2)} = i_2 \).

Do the same thing for every good \( j \): find a production vector \( x_l^{(j)} \) that yields a net output of one unit of good \( j \) and zero units of all other goods, so that we have \( (I - A_l)x_l^{(j)} = i_j \) for \( j = 1, 2, \ldots, n \). Stacking all of these vector equations next to each other, we obtain the following matrix equation:

\[
[(I - A_l)x_l^{(1)} \ (I - A_l)x_l^{(2)} \ \cdots \ (I - A_l)x_l^{(n)}] = [i_1 \ i_2 \ \cdots \ i_n] \tag{3a}
\]

\[
\Rightarrow (I - A_l)x_l^{(1)} \ x_l^{(2)} \ \cdots \ x_l^{(n)}] = [i_1 \ i_2 \ \cdots \ i_n] \tag{3b}
\]

Letting \( X_l = [x_l^{(1)} \ x_l^{(2)} \ \cdots \ x_l^{(n)}] \) which is an \( (n \times n) \) matrix of production plans for the capital goods that yield one unit of net output of each respective capital good, and observing that

\[
[i_1 \ i_2 \ \cdots \ i_n] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I, \tag{4}
\]

we have:

\[
(I - A_l)X_l = I
\]

\[
\Rightarrow X_l = A_lX_l + I \tag{5a}
\]

\[
(I - A_l)X_l = I
\]

\[
\Rightarrow X_l = A_lX_l + I \tag{5b}
\]

We could do the same kind of construction for the wage and luxury goods. Find a production vector \( x_l^{(n+1)} \) of capital goods that yields as net outputs the vector of capital input coefficients for the first wage/luxury good. Therefore, \( x_l^{(n+1)} \) would have to satisfy:

\(^1\) Recall from the “Matrix Algebra Tutorial” post that the prime superscript on the vector indicates “transpose.” The transpose of a matrix is obtained by interchanging rows and columns, so the first row of the old matrix becomes the first column of the new one, the second row becomes the second column, and so on. The transpose of a row vector is a column vector and vice-versa. So in this case, \( x_l \) refers to the column vector consisting of the values \( x_1, x_2, \ldots, x_n \).
\[(I - A_I)x_{II}^{(n+1)} = \begin{bmatrix} a_{1,n+1} \\ a_{2,n+1} \\ \vdots \\ a_{n,n+1} \end{bmatrix} \]

Similarly, for each wage good \(i = 1, 2, \ldots, m - n\), we find a production vector \(x_{II}^{(n+i)}\) of capital goods that yields as net outputs the vector of capital input coefficients for good \(i\). Thus:

\[(I - A_I)x_{II}^{(n+i)} = \begin{bmatrix} a_{1,n+i} \\ a_{2,n+i} \\ \vdots \\ a_{n,n+i} \end{bmatrix} \]

Stacking these vector equations side-by-side gives us the following matrix equation:

\[
(I - A_I)x_{II}^{(n+1)} \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} (I - A_I)x_{II}^{(n+2)} \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \cdots (I - A_I)x_{II}^{(m)} = \begin{bmatrix} a_{1,n+1} & a_{1,n+2} & \cdots & a_{1m} \\ a_{2,n+1} & a_{2,n+2} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,n+1} & a_{n,n+2} & \cdots & a_{nm} \end{bmatrix} \]

\[\Rightarrow (I - A_I)[x_{II}^{(n+1)} \ x_{II}^{(n+2)} \ \cdots \ x_{II}^{(m)}] = \begin{bmatrix} a_{1,n+1} & a_{1,n+2} & \cdots & a_{1m} \\ a_{2,n+1} & a_{2,n+2} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,n+1} & a_{n,n+2} & \cdots & a_{nm} \end{bmatrix} \]

Letting \(X_{II} = \begin{bmatrix} x_{II}^{(n+1)} & x_{II}^{(n+2)} & \cdots & x_{II}^{(m)} \end{bmatrix}\), which is an \((n \times (m - n))\) matrix of production plans for capital goods that yield as net outputs the capital input coefficient vectors for each respective wage / luxury good, and observing that the matrix on the right is simply \(A_{II}\), we have:

\[(I - A_I)X_{II} = A_{II} \]

\[\Rightarrow X_{II} = A_{II}X_{II} + A_{II} \]

Now thinking about what the second definition says, how would we formally define the amount of labor socially necessary for producing a unit of a capital good? Well, we have a production plan matrix \(X_I\) for capital goods that will yield net output of one unit of each respective capital good. If we count up the labor needed to produce the capital goods in each of these production plans, that should give us the labor socially necessary to produce a unit of each capital good. But the labor needed for these production plans would be calculated by just multiplying the labor input coefficient vector \(L_I\) by the production plan matrix \(X_I\). Therefore, letting \(M_I\) denote the vector of labor values of capital goods as defined in the second definition, we have:

\[M_I = L_I X_I \]

How do we define the amount of labor socially necessary for producing a unit of a wage or luxury good? We have developed a production plan matrix \(X_{II}\) for capital goods that yields as net outputs the capital input coefficient vectors for each respective wage or luxury goods. If we counted up the labor needed to produce the capital goods in each of these production plans (i.e.,
Then we would have accounted for all the labor needed to produce the capital goods that go into producing a unit of each wage or luxury good. But we would not be quite done because we also need to account for the labor directly needed to produce a unit of each wage or luxury good. Thus, we need to add in the labor input coefficient vector \( L_{II} \) for wage/luxury goods. Thus, letting \( M_{II} \) denote the vector of labor values of wage / luxury goods according to the second definition, we have:

\[
M_{II} = L_{I}X_{II} + L_{II} \tag{10b}
\]

Having developed the labor value vectors according to both definitions, our task now is to show that they are equal. That is, we need to show that \( L_{I} = M_{I} \) and \( L_{II} = M_{II} \). Observe that equation (1a) implies:

\[
\Lambda_{I}(I - A_{I}) = L_{I} \tag{11}
\]

and equation (5b) implies:

\[
(I - A_{I})X_{I} = I \tag{12}
\]

Now by (10a) and (11) we have:

\[
M_{I} = \Lambda_{I}(I - A_{I})X_{I} \tag{13a}
\]

\[
= \Lambda_{I}I \quad \text{by (12)} \tag{13b}
\]

\[
= \Lambda_{I} \tag{13c}
\]

Thus we have shown both definitions of labor value are the same for capital goods. To do so for wage and luxury goods, observe that (9b) implies:

\[
(I - A_{I})X_{II} = A_{II} \tag{14}
\]

Therefore, from (1b) and (14), we have:

\[
\Lambda_{II} = \Lambda_{I}(I - A_{I})X_{II} + L_{II} \tag{15a}
\]

\[
= L_{I}X_{II} + L_{II} \quad \text{by (11)} \tag{15b}
\]

\[
= M_{II} \quad \text{by (10b)} \tag{15c}
\]

Thus the labor values for wage and luxury goods are also the same according to both definitions.